IEOR 222

Financial Engineering

Exercises

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https://github.com/GennaroAlberto/ieor222-financial-engineering

Built October 31, 2026

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Part I

Probability and discrete-time markets

Chapter 1

Probability as information

Exercise 1.1 *(Algebras and σ-algebras)* [$\star$] An *algebra* on a set Ω is a collection of subsets that contains Ω and is closed under complements and *finite* unions.

(a) Show that on a finite set Ω every algebra is a σ-algebra.

(b) Let $\Omega = \mathbb{N}$ and let $\mathcal{A}$ be the collection of sets $A \subseteq \mathbb{N}$ such that A is finite or $\mathbb{N} \setminus A$ is finite. Show that $\mathcal{A}$ is an algebra but not a σ-algebra.

(c) Let $\Omega = \mathbb{R}$ and let $\mathcal{C}$ be the collection of sets $A \subseteq \mathbb{R}$ such that A is countable or $\mathbb{R} \setminus A$ is countable. Show that $\mathcal{C}$ is a σ-algebra, that $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$, and that the identity map $X(\omega) = \omega$ is not $\mathcal{C}$-measurable.

Exercise 1.2 *(Information in two tosses)* [$\star$] Take $n = 2$, $S_0 = 4$, $u = \frac{3}{2}$ and $d = \frac{1}{2}$, as in Theorem 1.4, and let $D = \mathbf{1}_{\{S_2 = 3\}}$ be the indicator that the two tosses differ.

(a) List the atoms of $\sigma(D)$ and of $\sigma(S_1, S_2)$, and all the events in $\sigma(D)$.

(b) For each of S_2/S_1, $|S_2 - S_1|$ and $\mathbf{1}_{\{S_2 \geq S_0\}}$, decide whether it is $\mathcal{F}_1$-measurable and whether it is $\sigma(S_2)$-measurable.

(c) For which $p \in (0,1)$ are D and S_1 independent under the p-coin? For the other values of p, compute $\mathbb{P}(S_1 = 6 \mid D = 1)$ and explain in words why learning that the tosses differ changes the odds of a first head.

(d) Under the $\frac{1}{2}$-coin, let $A = \{\omega_1 = H\}$, $B = \{\omega_2 = H\}$ and $C = \{\omega_1 = \omega_2\}$. Show that any two of these events are independent, but the three are not.

Exercise 1.3 *(Conditioning on four points)* [$\star$] Let $\Omega = \{a, b, c, d\}$ with $\mathcal{F} = 2^\Omega$ and $\mathbb{P}(a) = \frac{1}{8}$, $\mathbb{P}(b) = \frac{3}{8}$, $\mathbb{P}(c) = \frac{1}{3}$, $\mathbb{P}(d) = \frac{1}{6}$. Define

$$X = (1, 1, -1, -1), \qquad Y = (3, -1, 2, -1),$$

listing the values at a, b, c, d in this order.

(a) Give the atoms of $\sigma(X)$ and of $\sigma(Y)$.

(b) Compute $\mathbb{E}[Y \mid X]$, write it as a function of X, and check the partial-averaging property on every event of $\sigma(X)$.

(c) Compute $\mathbb{E}[X \mid Y]$ and write it as $\varphi(Y)$.

(d) Check the variance decomposition $\operatorname{Var}(Y) = \mathbb{E}\big[(Y - \mathbb{E}[Y \mid X])^2\big] + \operatorname{Var}(\mathbb{E}[Y \mid X])$.

(e) Compute $\mathbb{E}\big[\mathbb{E}[Y \mid X] \,\big|\, Y\big]$ and $\mathbb{E}\big[\mathbb{E}[Y \mid Y] \,\big|\, X\big]$. Are they equal? Why does the tower property not apply?

Exercise 1.4 *(The Doob–Dynkin lemma)* [⋆⋆] Let Y be a random variable and X a $\sigma(Y)$-measurable random variable. We show that $X = f(Y)$ for some Borel function $f : \mathbb{R} \to \mathbb{R}$. You may use that the pointwise lim sup of a sequence of Borel functions is Borel (as a function with values in $[-\infty, \infty]$).

(a) Prove the claim when X takes finitely many values $x_1, \dots, x_m$. *Hint:* each $\{X = x_i\}$ equals $\{Y \in B_i\}$ for some Borel B_i; make the B_i disjoint.

(b) Prove it in general, by approximating X with $X_j = \max\big(-j, \min(j, 2^{-j}\lfloor 2^j X \rfloor)\big)$.

(c) Deduce that for every integrable Z there is a Borel function φ with $\mathbb{E}[Z \mid Y] = \varphi(Y)$, and that φ is unique up to a Borel set B with $\mathbb{P}(Y \in B) = 0$. (This is how $\mathbb{E}[Z \mid Y = y] := \varphi(y)$ is defined when every event $\{Y = y\}$ has probability zero.)

Exercise 1.5 *(Conditioning with a density)* [⋆⋆] Let (X, Y) have a joint density $f(x, y)$ on $\mathbb{R}^2$, with $\mathbb{E}|X| < \infty$, and let $f_Y(y) = \int_{\mathbb{R}} f(x, y)\,\mathrm{d}x$ be the density of Y. Define

$$\varphi(y) = \frac{\int_{\mathbb{R}} x\, f(x, y)\,\mathrm{d}x}{f_Y(y)} \quad \text{if } 0 < f_Y(y) < \infty, \qquad \varphi(y) = 0 \ \text{ otherwise.}$$

You may use Fubini's theorem (e.g. Durrett, *Probability: Theory and Examples*, Section 1.7) and that φ is Borel.

(a) Show that $\varphi(Y) = \mathbb{E}[X \mid Y]$.

(b) Compute $\mathbb{E}[X \mid Y]$ when $f(x, y) = x + y$ on $[0, 1]^2$ and $f = 0$ elsewhere. Check that $\mathbb{E}\big[\mathbb{E}[X \mid Y]\big] = \mathbb{E}[X]$.

(c) Let (X, Y) be a centered Gaussian vector with $\operatorname{Var}(X) = \operatorname{Var}(Y) = 1$ and correlation $\rho \in (-1, 1)$. Show that the conditional law of X given $Y = y$ (the density $x \mapsto f(x, y)/f_Y(y)$) is $\mathcal{N}(\rho y, 1 - \rho^2)$, and deduce $\mathbb{E}[X \mid Y]$. Why does linear regression of X on Y lose nothing here (Theorem 1.21)?

Exercise 1.6 *(Conditional variance and random sums)* [⋆⋆] For $X \in L^2$ and a σ-algebra $\mathcal{G}$, define the conditional variance $\operatorname{Var}(X \mid \mathcal{G}) = \mathbb{E}\big[(X - \mathbb{E}[X \mid \mathcal{G}])^2 \,\big|\, \mathcal{G}\big]$.

(a) Show that $\operatorname{Var}(X \mid \mathcal{G}) = \mathbb{E}[X^2 \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{G}]^2$ and that $\operatorname{Var}(X) = \mathbb{E}[\operatorname{Var}(X \mid \mathcal{G})] + \operatorname{Var}(\mathbb{E}[X \mid \mathcal{G}])$.

(b) Let $A_1, A_2, \dots$ be a *countable* partition of Ω into events. Show that $\sigma(A_1, A_2, \dots)$ consists

of the unions of atoms, and that for integrable X,

$$\mathbb{E}[X \mid \sigma(A_1, A_2, \dots)] = \sum_{i:\, \mathbb{P}(A_i)>0} \frac{\mathbb{E}[X\mathbf{1}_{A_i}]}{\mathbb{P}(A_i)}\mathbf{1}_{A_i}.$$

(c) An insurer receives N claims in a year, of sizes $\xi_1, \xi_2, \dots$, and pays $S = \sum_{i=1}^{N} \xi_i$ (with $S = 0$ if $N = 0$). The ξ_i are i.i.d. with mean μ and variance s^2, and N, with values in $\mathbb{N}$ and $\mathbb{E}[N^2] < \infty$, is independent of $(\xi_1, \xi_2, \dots)$, meaning that N is independent of $(\xi_1, \dots, \xi_m)$ for every m. Show that $\mathbb{E}[S \mid N] = \mu N$ and $\mathrm{Var}(S \mid N) = s^2 N$, and deduce $\mathbb{E}[S]$ and $\mathrm{Var}(S)$. Evaluate them when N is Poisson with mean λ.

Exercise 1.7 *(Changing the coin)* [⋆⋆] On Ω_n, let $\mathbb{P}$ be the p-coin and $\mathbb{Q}$ the $\tilde{p}$-coin, with $p, \tilde{p} \in (0,1)$, $q = 1-p$, $\tilde{q} = 1-\tilde{p}$, so that $Z = \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ is given by Equation (1.8).

(a) Show that the density process is $Z_k = \mathbb{E}[Z \mid \mathcal{F}_k] = (\tilde{p}/p)^{H_k}(\tilde{q}/q)^{k-H_k}$, and check directly that $\mathbb{Q}(A) = \mathbb{E}[Z_k \mathbf{1}_A]$ for every atom A of $\mathcal{F}_k$.

(b) Using only Z and Bayes' formula (Theorem 1.24), show that $\mathbb{Q}(\omega_{k+1} = H \mid \mathcal{F}_k) = \tilde{p}$. Explain why this proves, once more, that under $\mathbb{Q}$ the tosses are independent with probability $\tilde{p}$ of heads.

(c) Compute $\mathbb{E}^{\mathbb{Q}}[S_n \mid \mathcal{F}_k]$ in two ways: by the independence lemma under $\mathbb{Q}$, and by Bayes' formula with $\mathbb{E}[ZS_n \mid \mathcal{F}_k]$ computed under $\mathbb{P}$.

(d) Let $r > -1$ with $d < 1 + r < u$. Find the unique $\tilde{p} \in (0,1)$ such that $\mathbb{E}^{\mathbb{Q}}[S_{k+1} \mid \mathcal{F}_k] = (1+r)S_k$ for every k. (This is the risk-neutral probability of Chapter 4.)

Exercise 1.8 *(Bayes' formula on a general space)* [⋆⋆⋆] Let $Z \geq 0$ with $\mathbb{E}[Z] = 1$, $\mathbb{Q}(A) = \mathbb{E}[Z\mathbf{1}_A]$, $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra, and X a random variable with $\mathbb{E}^{\mathbb{Q}}|X| = \mathbb{E}[Z|X|] < \infty$. Fix versions of $\mathbb{E}[Z \mid \mathcal{G}]$ and $\mathbb{E}[ZX \mid \mathcal{G}]$, and let $D = \{\mathbb{E}[Z \mid \mathcal{G}] > 0\}$. You may use the properties in Theorem 1.16 and the identity in Equation (1.4), with $A = \Omega$: $\mathbb{E}[WV] = \mathbb{E}\big[W\,\mathbb{E}[V \mid \mathcal{G}]\big]$ in $[0,\infty]$ for every $\mathcal{G}$-measurable $W \geq 0$ and every integrable $V \geq 0$.

(a) Show that $\mathbb{Q}(D^c) = 0$ and that $Z = 0$ almost surely on D^c.

(b) Show that $Y = \mathbf{1}_D\,\mathbb{E}[ZX \mid \mathcal{G}]/\mathbb{E}[Z \mid \mathcal{G}]$ is $\mathbb{Q}$-integrable and is a version of $\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{G}]$, the conditional expectation computed in $(\Omega, \mathcal{F}, \mathbb{Q})$.

(c) Show that if $Z > 0$ almost surely, then $\mathbb{P}(D^c) = 0$, so that Bayes' formula holds $\mathbb{P}$-almost surely. Give an example on a two-point space where $\mathbb{P}(D^c) > 0$ and the ratio in Equation (1.10) is $0/0$ on D^c.

Exercise 1.9 *(Importance sampling a rare event)* [⋆⋆] This is a computational exercise. Let $X \sim \mathcal{N}(0,1)$ under $\mathbb{P}$, and $\pi = \mathbb{P}(X > 4)$, a probability of the order of 3×10^{-5}.

(a) The plain Monte Carlo estimator averages $\mathbf{1}_{\{X_i>4\}}$ over M independent samples. Give its relative standard error, its standard deviation divided by π, for $M = 10^5$, and the probability that it returns 0.

(b) Let $\mathbb{Q}$ be the measure of Theorem 1.26 with parameter θ, so that $X \sim \mathcal{N}(\theta, 1)$ under $\mathbb{Q}$. Show that $\pi = \mathbb{E}^{\mathbb{Q}}\big[\mathbf{1}_{\{X>4\}} e^{-\theta X + \theta^2/2}\big]$, and that the second moment of the variable inside is $e^{\theta^2}\big(1 - N(4+\theta)\big)$, where N is the standard normal distribution function.

(c) Implement the estimator that averages $\mathbf{1}_{\{Y_i>4\}} e^{-\theta Y_i + \theta^2/2}$ over $M = 10^5$ independent samples $Y_i \sim \mathcal{N}(\theta, 1)$, for $\theta = 0, 1, \ldots, 6$. Report the estimates with 95% confidence intervals, and compare the empirical relative errors with those predicted by (b).

(d) Find numerically the θ that minimizes the variance in (b), and explain why both θ too small and θ too large are bad.

Chapter 2

Martingales and random walks

Exercise 2.1 *(Doob martingales and the Doob decomposition)* [⋆⋆] Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration.

(a) Let Y be an integrable random variable and $M_n = \mathbb{E}[Y \mid \mathcal{F}_n]$. Show that M is a martingale. Interpret M_n when Y is the discounted payoff of a derivative with maturity N and the expectation is taken under a risk-neutral probability.

(b) Let X be an adapted, integrable process. Show that there is exactly one way (up to almost sure equality) to write $X_n = X_0 + M_n + A_n$ for all n, with M a martingale, A predictable (A_n is $\mathcal{F}_{n-1}$-measurable for $n \geq 1$) and $M_0 = A_0 = 0$, and that $A_n = \sum_{k=1}^{n} \mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$. Show that A is nondecreasing if and only if X is a submartingale.

(c) Find A for $X_n = S_n$ (any p), and for $X_n = S_n^2$ with $p = \frac{1}{2}$ and $S_0 = 0$.

(d) Let $p = \frac{1}{2}$ and $S_0 = 0$. Show that for $X_n = |S_n|$, A_n is the number of visits of the walk to 0 before time n, that is, the number of $k \in \{0, \dots, n-1\}$ with $S_k = 0$. Deduce a formula for $\mathbb{E}|S_n|$ in terms of the probabilities $\mathbb{P}(S_k = 0)$. (This decomposition is a discrete Tanaka formula.)

Exercise 2.2 *(A discrete Itô isometry)* [⋆] Let M be a martingale with $\mathbb{E}[M_n^2] < \infty$ for every n, and H a bounded predictable process ($|H_n| \leq C_n$ for constants C_n). You may use that $\mathbb{E}[UV \mid \mathcal{G}] = U\,\mathbb{E}[V \mid \mathcal{G}]$ whenever U is $\mathcal{G}$-measurable and V and UV are integrable.

(a) Show that the increments of M are orthogonal, $\mathbb{E}\big[(M_k - M_{k-1})(M_j - M_{j-1})\big] = 0$ for $j < k$, and that $\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{k=1}^{n} \mathbb{E}\big[(M_k - M_{k-1})^2\big]$.

(b) Show that $\mathbb{E}\big[(H \cdot M)_n^2\big] = \sum_{k=1}^{n} \mathbb{E}\big[H_k^2 (M_k - M_{k-1})^2\big]$. What does this give when M is the symmetric random walk?

(c) Let $p = \frac{1}{2}$, $S_0 = 0$ and $H_k = S_{k-1}$. Why is H an admissible strategy for Theorem 2.8? Show that $(H \cdot S)_n = \frac{1}{2}(S_n^2 - n)$, and check (b) by computing both sides. You may use $\mathbb{E}[S_n^4] = 3n^2 - 2n$.

Exercise 2.3 *(Doubling at roulette)* [⋆] A gambler bets on red at roulette: each spin, independently, she wins her stake with probability $p = \frac{18}{37}$ and loses it with probability $q = \frac{19}{37}$. She uses

the doubling strategy of Theorem 2.13 with a bankroll of $2^m - 1$: she stakes $1, 2, 4, \dots$ and stops after her first win or after m consecutive losses, whichever comes first. Let G be her final gain.

(a) Find the law of G, and show that $\mathbb{E}[G] = 1 - (2q)^m$.

(b) Explain why $\mathbb{E}[G] \le 0$ follows from Theorem 2.8 without any computation.

(c) What happens to $\mathbb{E}[G]$ and to $\mathbb{P}(G > 0)$ as $m \to \infty$? Compute both for $m = 10$.

Exercise 2.4 *(No clairvoyance)* [⋆] Let σ and τ be stopping times for a filtration $(\mathcal{F}_n)$.

(a) Show that $\sigma \wedge \tau$, $\sigma \vee \tau$ and $\sigma + \tau$ are stopping times.

(b) Let $p = \frac{1}{2}$, $S_0 = 0$, and $\tau = \inf\{n \ge 0 : S_n = 1\}$. Show that $\tau - 1$ is not a stopping time. Show also that, for $N \ge 2$, the last visit to 0 up to time N, $\lambda_N = \max\{n \le N : S_n = 0\}$, is not a stopping time. What happens for $N = 1$?

(c) With $p = \frac{1}{2}$ and $S_0 = 0$, fix $N \ge 1$ and let ρ be the first time at which the walk attains its maximum over $\{0, \dots, N\}$. Show that ρ is not a stopping time in two ways: directly from the definition, and by computing the sign of $\mathbb{E}[S_\rho]$ and comparing with Theorem 2.14. Interpret this for a trader who must sell, by time N, a stock whose price is a martingale.

Exercise 2.5 *(Reaching a target in a favourable game)* [⋆⋆] Let $p > \frac{1}{2}$, $S_0 = 0$, fix an integer $b \ge 1$, and let $\sigma_b = \inf\{n \ge 0 : S_n = b\}$.

(a) Show that $\sigma_b < \infty$ almost surely. *Hint:* apply Theorem 2.19 to the walk $-S$.

(b) For $N \ge 1$ let τ_N be the exit time from $(-N, b)$. Show that $\mathbb{E}[\tau_N] \to \mathbb{E}[\sigma_b]$ as $N \to \infty$.

(c) Deduce from Theorem 2.18 that $\mathbb{E}[\sigma_b] = b/(p - q)$.

(d) Why could you not apply Wald's identity (Theorem 2.15) to σ_b directly? Once (c) is known, check it against Wald's identity. What happens as $p \downarrow \frac{1}{2}$, and how does it fit with Theorem 2.19?

Exercise 2.6 *(The variance of the duration)* [⋆⋆] Let $p = \frac{1}{2}$, $S_0 = 0$, fix an integer $b \ge 1$, and let τ be the exit time from $(-b, b)$.

(a) Find deterministic numbers c_n, with $c_0 = 0$, such that $S_n^4 - 6nS_n^2 + c_n$ is a martingale.

(b) Show that $\mathbb{E}\big[S_\tau^4 - 6\tau S_\tau^2 + c_\tau\big] = 0$. Justify every limit.

(c) Deduce that $\mathrm{Var}(\tau) = \frac{2}{3}b^2(b^2 - 1)$. Check the cases $b = 1$ and $b = 2$ by a direct argument.

Exercise 2.7 *(The law of an exit time)* [⋆⋆⋆] Let $p = \frac{1}{2}$, $S_0 = 0$, fix an integer $b \ge 1$, and let τ be the exit time from $(-b, b)$.

(a) Show that, for every $\theta \in \mathbb{R}$, $Y_n = \cosh(\theta S_n)/(\cosh\theta)^n$ is a martingale.

(b) Show that $\mathbb{E}\big[(\cosh\theta)^{-\tau}\big] = 1/\cosh(\theta b)$ for every $\theta \in \mathbb{R}$.

(c) Take $b = 2$. Compute $\mathbb{E}[s^\tau]$ for $s \in (0, 1]$, and identify the law of τ.

(d) Let $\theta \downarrow 0$ in (b) to show that $\mathbb{E}[\tau] = b^2$, without using Theorem 2.18. Justify the exchange of limit and expectation.

Exercise 2.8 *(Simulating the gambler's ruin)* [⋆⋆] This is a computational exercise. Let $p = 0.6$, $S_0 = 0$, and let τ be the first time the walk hits -10 or 10.

(a) Compute $\mathbb{E}[S_n]$ and $\mathrm{Var}(S_n)$. Use the notes to compute $\mathbb{P}(S_\tau = 10)$ and $\mathbb{E}[\tau]$ to four significant digits.

(b) Simulate 10^4 walks of 100 steps, plot two of them, and compare the sample mean of S_{100} with $\mathbb{E}[S_{100}]$, with a 95% confidence interval.

(c) For each simulated walk, find τ. Some walks may still be inside $(-10, 10)$ at time 100: how do you handle them? Estimate $\mathbb{P}(S_\tau = 10)$ and $\mathbb{E}[\tau]$ with 95% confidence intervals and compare with (a).

(d) Repeat (c) with $p = \frac{1}{2}$, first with the walks cut at 100 steps (counting a walk that has not exited as $\tau = 100$), then without cutting. Explain the difference, its sign, and why it is much larger than for $p = 0.6$.

Chapter 3

Arbitrage and static replication

Unless an exercise says otherwise, the market is frictionless, the stock pays no dividend, and the market is free of arbitrage in the sense of Theorem 3.1. Strikes, prices and payoffs are in dollars.

Exercise 3.1 *(Building payoffs from calls)* [⋆] Use Theorem 3.15.

(a) Replicate the *straddle*, which pays $|S_T - K|$ $(K > 0)$, with bonds, shares and calls. Replicate it again with one call and one put, and check with put–call parity that the two replications cost the same.

(b) A *collar* pays $\min\big(\max(S_T, 90), 120\big)$. Replicate it with bonds and calls, and again with one share, one put and one call. Compare the costs.

(c) A payoff equals 10 for $S_T \le 80$, decreases linearly to 0 at $S_T = 90$, increases linearly to 5 at $S_T = 100$, and equals 5 for $S_T \ge 100$. Replicate it and write its price in terms of B and call prices. Without computing anything, why is this price nonnegative?

(d) A *lookback call* with strike K pays $(M_T - K)^+$, where M_T is the highest price of the stock between 0 and T. Assuming it is traded and the market remains free of arbitrage, compare its price with $C(K)$.

Exercise 3.2 *(A known dividend)* [⋆] The stock now pays a known cash dividend $\delta > 0$ at a date $t_1 \in (0, T)$: whoever holds the share at t_1 receives δ, and a short seller must pay δ to the lender of the share. Besides the bond maturing at T (price B), a zero-coupon bond paying 1 at t_1 is traded, at price B_1. Portfolios are still static, but their payoffs now consist of a cash flow at t_1 and one at T. You may use Theorem 3.3 for such portfolios, with "$V \ge \tilde{V}$" meaning that the inequality holds at both dates.

(a) Show that the forward price for delivery at T is $F = (S_0 - \delta B_1)/B$.

(b) Show that $C(K) - P(K) = S_0 - \delta B_1 - BK$.

(c) Show that $(S_0 - \delta B_1 - BK)^+ \le C(K) \le S_0 - \delta B_1$.

(d) In Theorem 3.11, the lower bound for a call without dividends is at least the exercise value $S_0 - K$ when $B \le 1$. Show that this fails here: find conditions under which

$S_0 - \delta B_1 - BK < S_0 - K$ although $B_1 \le 1$ and $B \le 1$.

Exercise 3.3 *(A trainee's quote sheet)* [⋆⋆] The stock trades at $S_0 = 50$, the bond paying 1 at $T = 1$ at $B = 0.96$, and a trainee's program produces the following option prices for maturity T.

strike K	45	50	55
call	7.90	5.30	2.40
put	2.90	4.60	7.40

For each arbitrage below, give the portfolio, the cash collected at time 0, and the payoff at time T in every state.

(a) Check the calls alone against Theorem 3.10 and Theorem 3.12, and exploit every violation you find.

(b) Check the puts alone in the same way.

(c) Check put–call parity at each strike and exploit every violation.

(d) A *box spread* is long the 45/55 call spread and short the 45/55 put spread: long a call at 45, short a call at 55, short a put at 45, long a put at 55. Show that it pays 10 in every state. What should it cost? What does it cost at the quoted prices, and which discount factor does that imply?

(e) The trainee made two mistakes: one call price is a typo, and the puts were computed from parity with a wrong discount factor. Using (a)–(d), identify both, and correct the call price.

Exercise 3.4 *(What no-arbitrage leaves open)* [⋆] Fix a strike $K > 0$, and suppose that the market made of the bond, the stock and the call with strike K is free of arbitrage.

(a) Show that $\mathbb{P}(S_T > 0) > 0$, and deduce that $C(K) < S_0$. Show that $C(K) > 0$ if $\mathbb{P}(S_T > K) > 0$, and that $C(K) > S_0 - BK$ if $\mathbb{P}(S_T < K) > 0$.

(b) Suppose $S_0 > BK$. Show that $C(K) = S_0 - BK$ is possible only if $S_T \ge K$ almost surely, and that in that case the law of one price forces $C(K) = S_0 - BK$.

(c) Let c be a number with $(S_0 - BK)^+ < c < S_0$. Find $x > K$, a probability $\mathbb{P}$ under which S_T takes the two values 0 and x, each with positive probability, and a pricing measure (Theorem 3.5) under which the call costs c. What does this say about the bounds of Theorem 3.10?

(d) In your model of (c), what happens to x and to $\mathbb{Q}(S_T = x)$ as c increases to S_0? Interpret.

Exercise 3.5 *(Digital options)* [⋆⋆] A *digital call* with strike $K > 0$ pays 1 if $S_T > K$ and 0 otherwise; a *digital put* pays 1 if $S_T \le K$. Their prices are $D_c(K)$ and $D_p(K)$. Digitals and calls of every strike are traded, together with the bond and the stock.

(a) Show that $D_c(K) + D_p(K) = B$ and $0 \le D_c(K) \le B$. With $B = 0.98$, a dealer quotes a digital call at 1.01: build an arbitrage.

(b) For $0 < h < K$, show that

$$\frac{C(K) - C(K+h)}{h} \le D_c(K) \le \frac{C(K-h) - C(K)}{h}.$$

(c) Deduce that $-\partial^+ C(K) \le D_c(K) \le -\partial^- C(K)$. You may use that a convex function on an open interval has one-sided derivatives at every point, and that its difference quotients $(C(K+h) - C(K))/h$ decrease to $\partial^+ C(K)$ as $h \downarrow 0$, and $(C(K) - C(K-h))/h$ increase to $\partial^- C(K)$.

(d) Deduce that $D_c(K) = -C'(K)$ when C is differentiable at K. If there is a pricing measure $\mathbb{Q}$, express $D_c(K)$ in terms of $\mathbb{Q}$ and compare with Theorem 3.18.

(e) A trader has sold a digital call and wants to hold calls so that she never loses at T, whatever happens. Which portfolio of calls does (b) suggest, what does it cost, and what happens to this cost as $h \downarrow 0$?

Exercise 3.6 *(The log contract and the variance)* [⋆⋆] Let $\mathbb{Q}$ be a pricing measure for a market in which calls and puts of every strike $K > 0$ are traded, with $\mathbb{Q}(S_T > 0) = 1$, and let $F = S_0/B$. Assume that all the integrals of option prices that appear below are finite, so that Theorem 3.17 applies.

(a) Show that the claim paying $-\log(S_T/F)$ (the *log contract*) has price

$$L = \int_0^F \frac{P(K)}{K^2}\,\mathrm{d}K + \int_F^\infty \frac{C(K)}{K^2}\,\mathrm{d}K.$$

(b) Let $f : (0,\infty) \to \mathbb{R}$ be convex and twice continuously differentiable. Show, using Theorem 3.17 and the sign of option prices only, that the claim paying $f(S_T)$ costs at least $Bf(F)$. Which classical inequality have you proved, and under which measure?

(c) Show that the claim paying S_T^2 costs $BF^2 + 2\int_0^F P(K)\,\mathrm{d}K + 2\int_F^\infty C(K)\,\mathrm{d}K$, and deduce a formula for the variance of S_T under $\mathbb{Q}$ in terms of option prices.

(d) Suppose that under $\mathbb{Q}$, $\log S_T$ is normal with variance v^2. Show that its mean must be $\log F - v^2/2$, and that $L = Bv^2/2$. (This is why the log contract is the building block of variance swaps, in Chapter 15.)

Exercise 3.7 *(Reading a density off rounded quotes)* [⋆⋆] This is a computational exercise. Take $S_0 = 100$, $B = 0.97$, $F = S_0/B$, and suppose that the market's call prices are

$$C(K) = S_0\,N(d_+) - BK\,N(d_-), \qquad d_\pm = \frac{\log(F/K) \pm v^2/2}{v}, \qquad v = 0.25,$$

where N is the standard normal distribution function. (This is the Black–Scholes formula of Chapter 6; you may take it as given.) These prices come from a pricing measure under which S_T has the density

$$q_T(K) = \frac{1}{Kv\sqrt{2\pi}} \exp\Big(-\frac{\big(\log(K/F) + v^2/2\big)^2}{2v^2}\Big), \qquad K > 0.$$

(a) At $K = 100$ and $K = 130$, compute the butterfly estimate

$$\hat{q}_h(K) = \frac{C(K-h) - 2C(K) + C(K+h)}{Bh^2}$$

for $h \in \{20, 10, 5, 2, 1, 0.5\}$ and compare it with $q_T(K)$. How does the error behave as $h \to 0$? Explain the rate.

(b) Real quotes are rounded to the cent. Repeat (a) with the prices rounded to two decimals. What happens for small h, and why? Give a rough bound for the error caused by rounding, and choose a sensible h.

(c) Take $h = 0.1$ at $K = 130$: the rounded quotes give $\hat{q}_h(K) < 0$. Which relation of the notes do they violate? Is there an arbitrage at the rounded prices, and would you expect to be able to trade it?

Exercise 3.8 *(An arbitrage that depends on the model)* [⋆⋆⋆] Let $S_0 = 100$ and $B = 1$, and suppose that calls with strikes 100 and 110 are both quoted at 5. In this exercise a *model* is any probability $\mathbb{P}$ for $S_T \geq 0$; the prices are fixed.

(a) Check that the quotes satisfy every inequality of Theorem 3.10 and Theorem 3.12.

(b) Show that, whatever the model, the market made of the bond, the stock and the two calls has an arbitrage. (Distinguish the models with $\mathbb{P}(S_T > 100) > 0$ from the others.)

(c) Show that each of the two portfolios you used in (b) fails to be an arbitrage in some model.

(d) Show that no portfolio of the bond, the stock and the two calls is an arbitrage in every model. *Hint:* if θ were one, first use models in which S_T is constant to show that its payoff $V(x)$ is nonnegative for every $x \geq 0$. Then, for $x > 110$, use the model in which $S_T \in \{0, 100, x\}$ with probabilities chosen so that the expected payoffs of the bond, the stock and the call with strike 100 equal their prices, and let $x \to \infty$.

Chapter 4

The binomial model

Exercise 4.1 *(One period, with numbers)* [⋆] A stock costs $S_0 = 50$ today and will be worth 75 or 45 after one period. The interest rate is $r = 2\%$ per period.

(a) Check that there is no arbitrage and compute the risk-neutral probability q_u.

(b) Price and hedge a call and a put, both with strike $K = 55$. Describe each replicating portfolio (shares and cash), and check put–call parity.

(c) The call is quoted at 4.50. Construct an arbitrage and give its value at time 1 in both states.

(d) Suppose that the real-world probability of a rise is $p = 0.6$. Compute the expected rate of return $\mathbb{E}[V_1]/V_0 - 1$ under $\mathbb{P}$ of the stock, the call and the put, and compare them with r. Do the same under $\mathbb{Q}$. Explain the signs.

Exercise 4.2 *(A path-dependent claim)* [⋆⋆] In a two-period binomial model, $S_0 = 8$, $u = \frac{3}{2}$, $d = \frac{1}{2}$ and $r = \frac{1}{4}$. A *floating-strike lookback put* pays $V_2 = \max(S_0, S_1, S_2) - S_2$ at time 2.

(a) List the four paths with S_1, S_2 and V_2. Why is V_2 not a function of S_2?

(b) Compute V_1 and Δ_1 at both nodes at time 1, then V_0 and Δ_0.

(c) Start from $X_0 = V_0$, follow the self-financing equation Equation (4.1) along each path, and check that $X_2 = V_2$.

(d) Check that $V_0 = \mathbb{E}^{\mathbb{Q}}[(1+r)^{-2}V_2]$.

Exercise 4.3 *(State prices)* [⋆] Consider the N-period model with $d < 1 + r < u$, and let H_n be the number of heads among the first n tosses.

(a) Show that the price at time 0 of the claim $\mathbf{1}_{\{\varpi\}}$, which pays one dollar in the state ϖ and nothing otherwise, is $\pi(\varpi) = (1+r)^{-N}\mathbb{Q}(\varpi)$, and that every claim has price $V_0 = \sum_{\omega\in\Omega} \pi(\omega)V_N(\omega)$.

(b) By Equation (1.8) with $\tilde{p} = q_u$, the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$

is $Z = (q_u/p)^{H_N}(q_d/(1-p))^{N-H_N}$. Show that Z is a function of S_N, and write V_0 as an expectation under $\mathbb{P}$.

(c) Let $Z_n = \mathbb{E}[Z \mid \mathcal{F}_n]$, computed in Exercise 1.7(a). Show that $(1+r)^{-n} Z_n X_n$ is a $\mathbb{P}$-martingale for every portfolio X. *Hint:* Bayes' formula, Theorem 1.24.

(d) For which p is $Z \equiv 1$? Interpret this value of p in terms of the expected return of the stock under $\mathbb{P}$.

Exercise 4.4 *(American puts)* [⋆⋆] Consider the N-period model with $d < 1 + r < u$ and a strike $K > 0$.

(a) Show that if $r = 0$, the American put has the same price as the European put at every node. *Hint:* adapt the proof of Theorem 4.13(iv).

(b) Take $N = 2$, $S_0 = 8$, $u = \frac{3}{2}$, $d = \frac{1}{2}$, $r = \frac{1}{4}$ and $K = 8$. Compute the prices of the European and the American put at every node, find the optimal exercise time τ^*, and check that $U_0 = \mathbb{E}^{\mathbb{Q}}\big[(1+r)^{-\tau^*}(K - S_{\tau^*})^+\big]$.

(c) Let $r \geq 0$, and let C_0^A and P_0^A be the prices of the American call and put. Show that

$$S_0 - K \leq C_0^A - P_0^A \leq S_0 - (1+r)^{-N} K.$$

Hint: for the left inequality, write $(K - S_\tau)^+ = (S_\tau - K)^+ + K - S_\tau$ for a stopping time τ, and apply Theorem 4.12 to the discounted stock price.

Exercise 4.5 *(A dice game)* [⋆] You may roll a fair die up to n times. After each roll except the last you may stop and receive, in dollars, the number shown, or roll again; after the n-th roll you receive what it shows. Let X_k be the result of the k-th roll and $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$.

(a) For $n = 3$, find the fair value of the game (the expected gain under the best rule) and the best rule, by backward induction.

(b) Write the rule of (a) as a random time τ with values in $\{1, 2, 3\}$ and show that it is a stopping time. Check that the proof of Theorem 4.13(ii) goes through for this game, with $\mathbb{P}$ in place of $\mathbb{Q}$, $r = 0$, $G_k = X_k$ and time starting at 1, and deduce that the value of (a) is the largest $\mathbb{E}[X_\tau]$ over all stopping times τ with values in $\{1, 2, 3\}$.

(c) Let v_n be the value of the game with n rolls. Show that $v_1 = \frac{7}{2}$ and $v_{n+1} = \mathbb{E}\big[\max(X_1, v_n)\big]$, that v_n increases to 6, and find the smallest n with $v_n > 5$.

Exercise 4.6 *(The Snell envelope)* [⋆⋆⋆] In the setting of Theorem 4.13, write $Y_n = (1+r)^{-n} U_n$ and $\hat{G}_n = (1+r)^{-n} G_n$. Call $\tau \in \mathcal{T}_0$ *optimal* if $\mathbb{E}^{\mathbb{Q}}[\hat{G}_\tau] = U_0$.

(a) Show that Y is the smallest $\mathbb{Q}$-supermartingale that dominates $\hat{G}$: every $\mathbb{Q}$-supermartingale Z with $Z_n \geq \hat{G}_n$ for all n satisfies $Z_n \geq Y_n$ for all n.

(b) Show that τ is optimal if and only if $U_\tau = G_\tau$ and the stopped process $(Y_{n \wedge \tau})_{0 \leq n \leq N}$ is a $\mathbb{Q}$-martingale.

(c) Deduce that $\tau^* \leq \tau$ for every optimal τ.

(d) Let σ^* be the first $n < N$ with $U_n > \mathbb{E}^{\mathbb{Q}}[U_{n+1} \mid \mathcal{F}_n]/(1+r)$, and $\sigma^* = N$ if there is none. Show that σ^* is an optimal stopping time and that $\tau \le \sigma^*$ for every optimal τ.

Exercise 4.7 *(A trinomial market)* [⋆⋆] In a one-period trinomial model, $S_0 = 100$, the factors are $u = 1.3$, $m = 1.1$ and $d = 0.9$, and $r = 2\%$.

(a) Describe all risk-neutral measures.

(b) Find the interval of arbitrage-free prices of a call with strike 100.

(c) The call is quoted at 9.50. Construct an arbitrage. Do the same if it is quoted at 5.

(d) In a general one-period trinomial model with $d < 1 + r < u$, show that a claim $h(S_1)$ can be replicated by the stock and the money market if and only if $\mathbb{E}^{\mathbb{Q}}[h(S_1)]$ is the same for every risk-neutral $\mathbb{Q}$. Back to the numbers: which of the following can be replicated: a call with strike 90; a digital option paying 1 if $S_1 = 110$; a put with strike 100?

(e) The call with strike 100 is now traded at 7. A risk-neutral measure for the market with the money market, the stock and the call must also make the discounted call price a martingale. Show that there is exactly one, and that every claim can be replicated with the three assets. Find the price of the digital option of (d).

Exercise 4.8 *(Stopping times by brute force)* [⋆⋆] This is a computational exercise.

(a) Let $f(N)$ be the number of stopping times with values in $\{0, \dots, N\}$ in the N-period model. Show that $f(0) = 1$ and $f(N) = 1 + f(N-1)^2$, and compute $f(N)$ for $N \le 6$. *Hint:* either $\tau = 0$ on all of Ω, or $\tau \ge 1$; in the second case, look at $\tau - 1$ separately on $\{\omega_1 = H\}$ and on $\{\omega_1 = T\}$.

(b) Take $N = 4$, $S_0 = 100$, $u = 1.1$, $d = 0.9$, $r = 2\%$ and the American put with $K = 100$. Compute U_0 by backward induction. Then enumerate all stopping times, compute $\mathbb{E}^{\mathbb{Q}}[(1+r)^{-\tau}(K - S_\tau)^+]$ for each, and check that the largest value is U_0. How many stopping times attain it? Compare them with τ^* and with the time σ^* of Exercise 4.6.

(c) How does the cost of each method grow with N?

Part II

Brownian motion and the Black–Scholes formula

Chapter 5

Brownian motion

Exercise 5.1 *(Covariances)* [⋆] Let $0 < s < t$.

(a) Compute the correlation of W_s and W_t. What happens when $s/t \to 0$ and when $s/t \to 1$?

(b) Compute $\mathbb{E}[W_s^2 W_t^2]$ and $\mathrm{Cov}(W_s^2, W_t^2)$.

Exercise 5.2 *(Polynomial martingales)* [⋆] Let $0 \le s < t$.

(a) Compute $\mathbb{E}[W_t^3 \mid \mathcal{F}_s]$ and deduce that (W_t^3) is not a martingale.

(b) Show that $W_t^3 - 3tW_t$ is a martingale.

(c) Find the constant c for which $W_t^4 - 6tW_t^2 + ct^2$ is a martingale.

(d) Expand $\exp(\sigma W_t - \frac{\sigma^2}{2}t)$ in powers of σ. Explain, without full rigour, why the coefficients of σ, σ^2, σ^3, σ^4 should be martingales, and compare them with (b) and (c) and with Theorem 5.12.

Exercise 5.3 *(Conditioning on the future)* [⋆⋆] Let $0 < t < s$.

(a) Find the constant a such that $W_t - aW_s$ is independent of W_s.

(b) Deduce $\mathbb{E}[W_t \mid W_s]$ and $\mathrm{Var}(W_t \mid W_s)$.

(c) Interpret the result: given where the path is at time s, where do you expect it to have been at time t, and how uncertain is that guess at t close to 0 and to s?

Exercise 5.4 *(Correlated Brownian motions)* [⋆] Let W and B be independent Brownian motions, $\rho \in [-1, 1]$, and $Z_t = \rho W_t + \sqrt{1-\rho^2}\, B_t$.

(a) Show that Z is a Brownian motion.

(b) Compute $\mathrm{Cov}(W_t, Z_t)$ and the correlation of W_t and Z_t.

(c) Given a target correlation matrix for two assets, explain how you would simulate two correlated Brownian motions from two independent ones.

Exercise 5.5 *(Time inversion)* [⋆⋆] Define $X_0 = 0$ and $X_t = tW_{1/t}$ for $t > 0$.

(a) Show that every vector $(X_{t_1}, \dots, X_{t_m})$ is centered Gaussian with $\operatorname{Cov}(X_s, X_t) = \min(s, t)$.

(b) Show that $W_u/u \to 0$ almost surely as $u \to \infty$. *Hint:* first along the integers, using the strong law of large numbers; then control $Z_n = \sup_{u \in [n, n+1]} |W_u - W_n|$. The Z_n are identically distributed (why?), and you may use Doob's maximal inequality, $\mathbb{E}\big[\sup_{u \le 1} W_u^2\big] \le 4\,\mathbb{E}[W_1^2]$.

(c) Conclude that X, after redefinition on a null set if necessary, is a Brownian motion.

Exercise 5.6 *(Exit from an interval)* [⋆⋆] Let $a < 0 < b$ and $\tau = \inf\{t \ge 0 : W_t \notin (a, b)\}$. A random time τ is a *stopping time* if $\{\tau \le t\} \in \mathcal{F}_t$ for every t; exit times of a continuous adapted process from an open interval are stopping times. You may use: (i) if M is a continuous martingale and τ a stopping time, then $(M_{t \wedge \tau})_{t \ge 0}$ is a martingale; (ii) the monotone and dominated convergence theorems.

(a) Using the martingale $W_t^2 - t$, show that $\mathbb{E}[\tau] < \infty$, and in particular that $\tau < \infty$ almost surely.

(b) Using the martingale W_t, compute $\mathbb{P}(W_\tau = b)$.

(c) Show that $\mathbb{E}[\tau] = -ab$.

(d) For $a = -b$, show with Brownian scaling (Theorem 5.9 and Theorem 5.10) that τ has the same law as $b^2 \tau_1$, where τ_1 is the exit time from $(-1, 1)$. Check that this agrees with (c).

Exercise 5.7 *(All the p-variations)* [⋆⋆] Let Π_n be the uniform partition of $[0, t]$ into n intervals, and $c_p = \mathbb{E}|N|^p$ for $N \sim \mathcal{N}(0, 1)$.

(a) Show that $\mathbb{E}[V_p(W; \Pi_n)] = c_p\, t^{p/2} n^{1 - p/2}$, and discuss its limit as $n \to \infty$ for $p < 2$, $p = 2$ and $p > 2$.

(b) Show that $V_p(W; \Pi_n) \to 0$ in probability for every $p > 2$. *Hint:* compare with V_2 as in the proof of Theorem 5.17.

(c) Show that $V_p(W; \Pi_n) \to \infty$ in probability for every $p < 2$.

Exercise 5.8 *(Simulating an exit time)* [⋆⋆] This is a computational exercise. Take $a = -0.8$ and $b = 0.5$, as in Exercise 5.6.

(a) Simulate Brownian paths on a grid of step h and estimate $\mathbb{E}[\tau]$ by the first grid time at which the path leaves (a, b). Report a 95% confidence interval.

(b) Repeat for $h = 10^{-2}, 10^{-3}, 10^{-4}$. Is your estimate biased? In which direction, and why? How does the bias appear to scale with h?

(c) Compare with the exact value from Exercise 5.6.

Chapter 6

From the binomial model to Black–Scholes

Exercise 6.1 *(Rules of thumb)* [⋆] A stock trades at $S_0 = 80$, the interest rate is $r = 3\%$, the volatility is $\sigma = 25\%$, and options expire in two months ($T = 1/6$). A calculator or a few lines of code are enough.

(a) Compute the at-the-money-forward strike, and the exact price, delta and vega of the call with this strike. Compare each with the approximation of Theorem 6.16 that uses $1/\sqrt{2\pi}$, and with the rounded rule that uses 0.4 and 0.2.

(b) The volatility rises from 25% to 26%. Estimate the change in the call price with the vega and with the rule $0.4\, S_0\sqrt{T}$, and compare with the exact change.

(c) Now take the strike $K = S_0$. Compute the call and put prices and compare both with $0.4\, S_0\sigma\sqrt{T}$. Using the derivatives with respect to K in Theorem 6.14, show that for every $r \geq 0$ the call with strike S_0 costs at least as much as the call with strike $S_0 e^{rT}$, and at most $S_0(1 - e^{-rT})$ more. State and prove the corresponding statement for puts.

Exercise 6.2 *(Power claims and a second martingale)* [⋆] Let $p \in \mathbb{R}$.

(a) Show that the price at time $t \leq T$ of the claim paying S_T^p at T is

$$V_t = S_t^p \exp\Big((p-1)\Big(r + \frac{p\sigma^2}{2}\Big)(T-t)\Big).$$

Check the cases $p = 0$ and $p = 1$, and give the delta $\partial V_t/\partial S_t$.

(b) Show that $(e^{-rt}S_t^p)_{t\geq 0}$ is a $\mathbb{Q}$-martingale if and only if $p = 1$ or $p = -2r/\sigma^2$. What happens when $r = 0$?

(c) Let $r > 0$ and $\beta = 2r/\sigma^2$. Show that $v(x) = x^{-\beta}$ solves $rv = rxv' + \frac{\sigma^2 x^2}{2}v''$ on $(0,\infty)$, the equation Equation (6.7) for a price that does not depend on time, and relate this to (b). Which other power of x solves it?

Exercise 6.3 *(Forward-start options)* [⋆⋆] Let $0 < \theta < T$. A *forward-start call* pays $(S_T - S_\theta)^+$ at time T: it is an at-the-money call whose strike is fixed at the future date θ, as in some employee stock option plans.

(a) For $\theta \le t < T$, show that its price is $c(t, S_t)$ computed with the strike $K = S_\theta$. *Hint:* the strike is random; apply Theorem 1.19 with $X = (S_t, S_\theta)$.

(b) Show that for every $x > 0$ and $t < T$, the call function computed with strike $K = x$ and evaluated at x is x times its value at $x = K = 1$. Deduce that for $0 \le t \le \theta$ the price is $V_t = \alpha S_t$, with

$$\alpha = N(\hat{d}_+) - e^{-r(T-\theta)} N(\hat{d}_-), \qquad \hat{d}_\pm = \frac{(r \pm \sigma^2/2)\sqrt{T-\theta}}{\sigma}.$$

(c) Give the delta and the gamma for $t < \theta$, and a hedge of the claim on $[0, \theta]$ that never needs rebalancing. Why is the gamma zero before θ and positive after?

(d) Take $S_0 = 100$, $r = 5\%$, $\sigma = 20\%$, $\theta = \frac{1}{2}$ and $T = 1$. Compute V_0 and check it by Monte Carlo simulation of (S_θ, S_T).

Exercise 6.4 *(Chooser options)* [⋆⋆] Let $0 < m < T$ and $K > 0$. A *chooser option* lets its holder decide at time m whether it becomes a call or a put, both with strike K and maturity T. Let C_t and P_t be the prices of that call and that put.

(a) Explain why the chooser is worth $\max(C_m, P_m)$ at time m, so that its price at time 0 is that of a claim paying $\max(C_m, P_m)$ at time m.

(b) Show that $\max(C_m, P_m) = C_m + \left(Ke^{-r(T-m)} - S_m\right)^+$.

(c) Deduce that the chooser costs as much as a call with strike K and maturity T plus a put with strike $Ke^{-r(T-m)}$ and maturity m, and give its price, delta and gamma at time 0 in terms of N, N' and the appropriate $d_\pm$.

(d) Take $S_0 = K = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 1$ and $m = \frac{1}{2}$. Compare the prices of the chooser, of the call, and of a *straddle* (a call plus a put, both with strike K and maturity T). Explain the ordering, and what happens as $m \downarrow 0$ and as $m \uparrow T$.

Exercise 6.5 *(Hedging a digital option)* [⋆⋆] Let $K > 0$ and let $D(t, x) = e^{-r\tau} N\big(d_-(\tau, x)\big)$ be the price function of the digital call of Theorem 6.13, for $\tau = T - t > 0$.

(a) Compute the delta $\partial_x D$ and the gamma $\partial_{xx} D$. Show that the delta is positive and that the gamma changes sign at $x^* = Ke^{-(r+\sigma^2/2)\tau}$.

(b) Show that the largest delta over $x > 0$ is $1/(K\sigma\sqrt{2\pi\tau})$. What happens to the delta as $\tau \downarrow 0$, at $x = K$ and at $x \neq K$? Compare with the gamma of a call (Theorem 6.15).

(c) A dealer has sold one million digital calls, each paying \$1, with $K = 100$, on a stock at $S_t = 100$ with $\sigma = 20\%$ and $r = 0$. How many shares must she hold to be delta-neutral one year before maturity? One day before ($\tau = 1/365$)? Compare the value of these shares with the largest possible payout, and explain why a move of the stock by \$1 in either direction changes her position so much.

(d) How does the call spread of Exercise 3.5(e) avoid the problem?

Exercise 6.6 *(Long maturities and negative rates)* [⋆⋆] Fix $x > 0$, $K > 0$ and $\sigma > 0$, and let $C(T) = c(0, x)$ be the price at time 0 of the call with maturity T when the stock is at x. You may use the bound $N(y) \le N'(y)/|y|$ for $y < 0$.

(a) Let $r \ge 0$, so that C is strictly increasing in T (Theorem 6.15(a)). Find its limits as $T \downarrow 0$ and as $T \to \infty$.

(b) Let $r < 0$. Show that $C'(T) < 0$ for x large enough, with T fixed.

(c) Let $r < 0$. Show that $\lim_{T\to\infty} C(T)$ is x, $x/2$ or 0 according as $r > -\sigma^2/2$, $r = -\sigma^2/2$ or $r < -\sigma^2/2$. *Hint:* first show that $Ke^{-rT}N(d_-) \to 0$, using Equation (6.6).

(d) Show that when $r < 0$ a European call can be worth strictly less than its exercise value $x - K$. Which hypothesis of Theorem 4.13(iv) does this example violate, and what does it suggest about American calls when rates are negative?

Exercise 6.7 *(The binomial hedge converges to the Black–Scholes delta)* [⋆⋆⋆] Consider the trees Equation (6.1) for $n > r^2T/\sigma^2$, a call with strike $K > 0$ and maturity T, and let $\Delta_0^{(n)}$ be the number of shares held at time 0 in its replicating portfolio, given by Equation (4.8).

(a) Let $F_n(y)$ be the price of the call at time δ_n in the n-period tree, as a function of the stock price y at that time; it is the price in an $(n-1)$-period tree with the same u_n, d_n and r_n, started at y. Show that $\Delta_0^{(n)} = (F_n(u_nS_0) - F_n(d_nS_0))/((u_n - d_n)S_0)$ and that F_n is convex on $(0, \infty)$.

(b) Show that $F_n(y) \to c(0, y)$ for every $y > 0$, where $c(0, \cdot)$ is the Black–Scholes call function with maturity T. *Hint:* Theorem 6.3 with $k_n = n - 1$, and the proof of Theorem 6.4.

(c) Let f_n and f be convex functions on an open interval I with $f_n \to f$ pointwise on I, let f be differentiable at $x \in I$, and let $a_n \le x \le b_n$ with $a_n < b_n$ and $a_n, b_n \to x$. Show that $(f_n(b_n) - f_n(a_n))/(b_n - a_n) \to f'(x)$. You may use that the slope of a chord of a convex function does not decrease when either end point moves to the right.

(d) Conclude that $\Delta_0^{(n)} \to N(d_+(T, S_0))$.

(e) For $S_0 = K = 100$, $T = 1$, $r = 5\%$ and $\sigma = 20\%$, compute $\Delta_0^{(n)}$ for $n = 10$, 11, 100, 101, 1000 and 1001, and compare with $N(d_+)$.

Exercise 6.8 *(Speeding up the tree)* [⋆⋆] This is a computational exercise. Take $S_0 = 100$, $T = 1$, $r = 5\%$, $\sigma = 20\%$, and let C_n be the call price in the n-period tree Equation (6.1) and C the Black–Scholes price. You may use `binomial_call` and `call` from `code/src/ieor222/blackscholes.py`.

(a) For $K = 100$ and $K = 110$, tabulate $e_n = C_n - C$ for $n = 50, 51, 100, 101, \ldots, 800, 801$. For $K = 100$, estimate the rate of convergence separately for even and odd n. What do you observe for $K = 110$?

(b) Average consecutive trees: $\bar{C}_n = \frac{1}{2}(C_n + C_{n+1})$. How much does this help for $K = 100$, and

for $K = 110$? Explain with Theorem 6.7.

(c) *Richardson extrapolation.* If $e_n = a/n + o(1/n)$ along a sequence of n, then $2C_{2n} - C_n$ has error $o(1/n)$. Try it along even n for $K = 100$ and for $K = 110$. When does it work, and why?

(d) *Smoothing.* Replace the payoff at the last step by the Black–Scholes price over one period: set $\tilde{V}_{n-1} = c(T - \delta_n, S_{n-1})$ at each node at time $n - 1$ and run the backward induction Equation (4.7) from there. Show that the resulting price $\tilde{C}_n$ is the expectation under $\mathbb{Q}_n$ of a discounted *smooth* function of S_{n-1}, repeat (a) and (c) for $\tilde{C}_n$, and comment.

Part III

Stochastic calculus

Chapter 7

The Itô integral

Exercise 7.1 *(Using the isometry)* [⋆]

(a) Let $p > 0$, and let $H_t = |W_t|^{-p}$ if $W_t \neq 0$ and $H_t = 0$ otherwise. For which values of p is H in $\mathbb{H}^2$?

(b) Compute $\operatorname{Var}\left(\int_0^T e^{W_t}\,\mathrm{d}W_t\right)$ and $\operatorname{Cov}\left(W_T, \int_0^T e^{W_t}\,\mathrm{d}W_t\right)$.

(c) Let $I_t = \int_0^t W_s\,\mathrm{d}W_s$. Compute $\mathbb{E}[[I,I]_T]$, $\operatorname{Var}(I_T)$ and $\operatorname{Var}\left([I,I]_T\right)$. You may use $\mathbb{E}[W_s^2 W_t^2] = 2s^2 + st$ for $s \leq t$ (Exercise 5.1). What would $\operatorname{Var}([I,I]_T)$ be for a deterministic integrand?

Exercise 7.2 *(Adaptedness cannot be dropped)* [⋆] Let $\Pi = \{t_j\}$ be partitions of $[0,T]$ with $\|\Pi\| \to 0$, and $\Delta_j = W_{t_{j+1}} - W_{t_j}$.

(a) Compute $\sum_j W_T \Delta_j$. If the integral of the process $H_t = W_T$, which is not adapted, were defined as the limit of these sums, what would its mean and its second moment be? Which statements of Theorem 7.7 would fail?

(b) Let R^Π be the process equal to $W_{t_{j+1}}$ on $[t_j, t_{j+1})$. Show that $\|R^\Pi - W\| \to 0$, although the sums $\sum_j W_{t_{j+1}} \Delta_j$ do not converge to $\int_0^T W_t\,\mathrm{d}W_t$ (Theorem 7.11). Why does this not contradict Equation (7.5)? Compare $\mathbb{E}\big[\big(\sum_j (W_{t_{j+1}} - W_{t_j})\Delta_j\big)^2\big]$ with $\|R^\Pi - W^\Pi\|^2$, where W^Π equals W_{t_j} on $[t_j, t_{j+1})$.

(c) Interpret (a) and (b) for a trader.

Exercise 7.3 *(Integration by parts for deterministic integrands)* [⋆⋆] Let $f : [0,T] \to \mathbb{R}$ be continuously differentiable.

(a) Show that for each $t \in [0,T]$, almost surely,

$$\int_0^t f(s)\,\mathrm{d}W_s = f(t)W_t - \int_0^t f'(s)W_s\,\mathrm{d}s,$$

where the last integral is an ordinary Riemann integral, computed path by path. *Hint:* Theorem 7.9, and summation by parts.

(b) Deduce that $\int_0^T W_s\,\mathrm{d}s$ has law $\mathcal{N}(0, T^3/3)$, and compute its covariance with W_T.

(c) In a Gaussian short-rate model (a special case of the Ho–Lee model of Chapter 14), the short rate is $r_t = r_0 + \theta t + \sigma W_t$ with constants $r_0, \theta \in \mathbb{R}$ and $\sigma > 0$, where W is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. The price at time 0 of a zero-coupon bond paying 1 at time T is $P(0,T) = \mathbb{E}^{\mathbb{Q}}\big[\exp\big(-\int_0^T r_t\,\mathrm{d}t\big)\big]$. Compute $P(0,T)$ and the yield $y(T) = -\frac{1}{T}\log P(0,T)$. What happens to $y(T)$ as $T \to \infty$, and why?

Exercise 7.4 *($\int W^2\,\mathrm{d}W$ by hand)* [⋆⋆] Let $\Pi = \{t_j\}$ be partitions of $[0,T]$ with $\|\Pi\| \to 0$, $\Delta_j = W_{t_{j+1}} - W_{t_j}$ and $h_j = t_{j+1} - t_j$.

(a) Check that $H_t = W_t^2$ satisfies the hypotheses of Theorem 7.4.

(b) Show that $\sum_j W_{t_j}^2 \Delta_j = \frac{1}{3}W_T^3 - \sum_j W_{t_j}\Delta_j^2 - \frac{1}{3}\sum_j \Delta_j^3$.

(c) Show that $\sum_j \Delta_j^3 \to 0$ and $\sum_j W_{t_j}\Delta_j^2 \to \int_0^T W_t\,\mathrm{d}t$ in L^2. You may use $\mathbb{E}[N^6] = 15$ for $N \sim \mathcal{N}(0,1)$.

(d) Conclude that $\int_0^T W_t^2\,\mathrm{d}W_t = \frac{1}{3}W_T^3 - \int_0^T W_t\,\mathrm{d}t$. Using Exercise 5.2, deduce that $tW_t - \int_0^t W_s\,\mathrm{d}s$ is a martingale, and write it as a Wiener integral (Exercise 7.3).

Exercise 7.5 *(A random integrand with a Gaussian integral)* [⋆⋆] Let $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x \le 0$, and let $I_t = \int_0^t \operatorname{sgn}(W_s)\,\mathrm{d}W_s$ for $t \ge 0$ (computed on any horizon $T \ge t$; by Theorem 7.8 the horizon does not matter).

(a) Check that the integral is defined, and compute $\operatorname{Cov}(I_s, I_t)$ and $[I,I]_t$.

(b) Lévy's characterization (Theorem 8.20) states that a martingale $(M_t)_{t\ge0}$ with continuous paths, $M_0 = 0$ and $[M,M]_t = t$ for every $t \ge 0$ is a Brownian motion. Deduce that I is a Brownian motion.

(c) Show that $\mathbb{E}[I_t W_t] = 0$, and compute $\mathbb{E}[I_t W_t^2]$ with the help of Theorem 7.11. Are I_t and W_t independent? Is (I_t, W_t) a Gaussian vector?

(d) Does this contradict Theorem 7.10?

Exercise 7.6 *(Simple processes are dense)* [⋆⋆⋆] This exercise proves Theorem 7.5. Let $H \in \mathbb{H}^2$. You may use two facts: (F) if $f \in L^2([0,T])$ and $f_n(t) = n\int_{(t-1/n)\vee 0}^t f(s)\,\mathrm{d}s$, then $\int_0^T (f_n - f)^2\,\mathrm{d}t \to 0$; and adapted processes with continuous paths are progressively measurable.

(a) Let $H^{(N)} = \max(-N, \min(N, H))$. Show that $H^{(N)} \in \mathbb{H}^2$ and $\|H^{(N)} - H\| \to 0$ as $N \to \infty$.

(b) Suppose that $|H| \le N$, and let $G_t^n = n\int_{(t-1/n)\vee 0}^t H_s\,\mathrm{d}s$. Show that G^n is adapted, has continuous paths and is bounded by N, and that $\|G^n - H\| \to 0$. Where do you use that H is progressively measurable, and not only adapted?

(c) Conclude, with Theorem 7.4.

(d) Why would the averages $n\int_t^{(t+1/n)\wedge T} H_s\,\mathrm{d}s$ over the near future not do?

Exercise 7.7 *(Wald's identities for Brownian motion)* [⋆⋆] Stopping times are as in Section 7.7.

(a) Let τ be a stopping time with values in $[0,T]$. Deduce from Theorem 7.15 that $W_\tau = \int_0^T \mathbf{1}_{[0,\tau]}(s)\,\mathrm{d}W_s$, and then Wald's identities $\mathbb{E}[W_\tau] = 0$ and $\mathbb{E}[W_\tau^2] = \mathbb{E}[\tau]$. Show also that $\mathbb{E}[(W_\tau - W_\rho)^2] = \mathbb{E}[\tau - \rho]$ for stopping times $\rho \le \tau$.

(b) Let $\sigma = \inf\{t \ge 0 : W_t = 1\}$, which is finite almost surely (Brownian motion reaches every level: let $a \to -\infty$ in Exercise 5.6). For every $T > 0$, $\sigma \wedge T$ is a stopping time with values in $[0,T]$. Show that $\mathbb{E}[W_{\sigma\wedge T}] = 0$ for every T, while $W_\sigma = 1$, and that $\mathbb{E}[\sigma] = \infty$.

(c) Interpret (b) for a trader who holds one unit of an asset with price W until her gain reaches 1, and compare with Theorem 2.19.

Exercise 7.8 *(Holding the asset when its price is positive)* [⋆⋆] This is a computational exercise. Take $T = 1$. A trader holds one unit of an asset with price W_t when $W_t > 0$, and none otherwise: $H_t = \mathbf{1}_{\{W_t > 0\}}$, with gains $I = \int_0^1 H_t\,\mathrm{d}W_t$.

(a) Show that $H \in \mathbb{H}^2$, and compute $\mathbb{E}[I]$, $\mathbb{E}[I^2]$ and $\mathbb{E}[IW_1]$.

(b) H is not continuous in mean square at $t = 0$, so Theorem 7.4 does not apply as stated. Let Π be the uniform partition with step $h = 1/n$, and H^Π as in the lemma. Using $\mathbb{P}(W_s > 0, W_t \le 0) = \frac{1}{2\pi}\arccos\sqrt{s/t}$ for $0 < s < t$, show that $\|H^\Pi - H\|^2 \le h/2 + \sqrt{h}$, so that the Itô sums still converge to I in L^2.

(c) Simulate 10^5 paths on uniform grids with $n = 10$, 100 and 1000 steps. Estimate $\mathbb{E}[I]$, $\mathbb{E}[I^2]$ and $\mathbb{E}[IW_1]$ from the Itô sums, with 95% confidence intervals, and compare with (a). Explain any discrepancy at $n = 10$.

(d) Compute $\mathbb{E}[W_1^+]$, and estimate $\mathbb{E}[W_1^+ - I]$ from your simulation. Since the derivative of $x \mapsto x^+$ is $\mathbf{1}_{\{x>0\}}$ for $x \neq 0$, ordinary calculus would suggest $W_1^+ = \int_0^1 \mathbf{1}_{\{W_t>0\}}\,\mathrm{d}W_t$. Why can this not hold? Compare with Theorem 7.11.

Chapter 8

Itô's formula

Exercise 8.1 *(Moments from Itô's formula)* [⋆] You may use that W_s has finite moments of all orders.

(a) For an integer $k \geq 2$, show that $\mathbb{E}[W_t^k] = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[W_s^{k-2}]\,\mathrm{d}s$. Say why the stochastic integral that appears has mean zero. Deduce that $\mathbb{E}[W_t^{2m}] = (2m-1)(2m-3)\cdots 3\cdot 1\cdot t^m$ and $\mathbb{E}[W_t^{2m+1}] = 0$ for $m \in \mathbb{N}$.

(b) Fix $u \in \mathbb{R}$ and let $\varphi(t) = \mathbb{E}[\cos(uW_t)]$ and $\psi(t) = \mathbb{E}[\sin(uW_t)]$. Show that $\varphi(t) = 1 - \frac{u^2}{2}\int_0^t \varphi(s)\,\mathrm{d}s$ and $\psi(t) = -\frac{u^2}{2}\int_0^t \psi(s)\,\mathrm{d}s$, and deduce that $\mathbb{E}[e^{iuW_t}] = e^{-u^2t/2}$.

Exercise 8.2 *(Itô decompositions)* [⋆] Fix $T > 0$. For each process, write its Itô decomposition and decide whether it is a martingale on $[0,T]$, checking the integrability condition that you use.

(a) $X_t = tW_t^2$.

(b) $Y_t = (W_t - t)\,e^{W_t - t/2}$.

(c) Explain (b) by differentiating $Z_t^\sigma = \exp(\sigma W_t - \frac{\sigma^2}{2}t)$ with respect to σ. Which martingale does the second derivative at $\sigma = 0$ give?

(d) $U_t = e^{W_t^2}$. Show that its diffusion coefficient Δ_s satisfies $\mathbb{E}[\Delta_s^2] = 4s(1-4s)^{-3/2}$ for $s < \frac{1}{4}$ and $\mathbb{E}[\Delta_s^2] = \infty$ for $s \geq \frac{1}{4}$, and deduce that Δ is in $\mathbb{H}^2$ on $[0,T]$ if and only if $T < \frac{1}{4}$. For which t is $\mathbb{E}[U_t]$ finite?

Exercise 8.3 *(A martingale has no drift)* [⋆⋆] Let $T > 0$ and $X_t = x_0 + \int_0^t \Theta_s\,\mathrm{d}s + \int_0^t \Delta_s\,\mathrm{d}W_s$, where Δ is in $\mathbb{H}^2$ on $[0,T]$ and $\mathbb{E}\big[\big(\int_0^T |\Theta_s|\,\mathrm{d}s\big)^2\big] < \infty$.

(a) Show that if $\Theta = 0$ for $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-almost every (t,ω), then X is a martingale on $[0,T]$.

(b) Conversely, let X be a martingale on $[0,T]$. Show that $A_t = \int_0^t \Theta_s\,\mathrm{d}s$ is a square-integrable martingale on $[0,T]$, and that $\mathbb{E}[A_t^2] = \mathbb{E}\big[\sum_j (\delta_j A)^2\big]$ for every partition of $[0,t]$. Deduce that $A \equiv 0$ on $[0,T]$, and that $\Theta = 0$ for $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-almost every (t,ω). *Hint:* bound $\sum_j (\delta_j A)^2$ by an integrable variable that does not depend on the partition.

(c) Let $f \in C^{1,2}([0,T]\times\mathbb{R})$ and $Lf = \partial_t f + \frac{1}{2}\partial_{xx} f$, and assume $\mathbb{E}\int_0^T \partial_x f(s, W_s)^2\,\mathrm{d}s < \infty$

and $\mathbb{E}\big[\big(\int_0^T |Lf(s,W_s)|\,\mathrm{d}s\big)^2\big] < \infty$. Show that $(f(t,W_t))_{t\le T}$ is a martingale if and only if $Lf = 0$ on $[0,T]\times\mathbb{R}$.

Exercise 8.4 *(The price of one stock in units of another)* [⋆⋆] Let (W^1, W^2) be a two-dimensional Brownian motion, $\rho \in [-1,1]$, $B^1 = W^1$, $B^2 = \rho W^1 + \sqrt{1-\rho^2}\,W^2$, and, for $i = 1,2$, $S^i_t = S^i_0 \exp\big(\sigma_i B^i_t + (\mu_i - \frac{\sigma_i^2}{2})t\big)$ with $S^i_0 > 0$, $\sigma_i > 0$ and $\mu_i \in \mathbb{R}$.

(a) The notes give $\mathrm{d}S^i_t$ (Theorem 8.8) and $\mathrm{d}[S^1,S^2]_t = \rho\sigma_1\sigma_2 S^1_t S^2_t\,\mathrm{d}t$ (the paragraph on correlated Brownian motions after Theorem 8.17). Compute $\mathrm{d}(1/S^2_t)$, and show that $U = S^1/S^2$ satisfies

$$\mathrm{d}U_t = U_t\Big[(\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2)\mathrm{d}t + \sigma_1\,\mathrm{d}B^1_t - \sigma_2\,\mathrm{d}B^2_t\Big].$$

(b) Let $\bar\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$ and assume $\bar\sigma > 0$. Show that $\bar B = (\sigma_1 B^1 - \sigma_2 B^2)/\bar\sigma$ is a Brownian motion and that U is a geometric Brownian motion with volatility $\bar\sigma$ driven by $\bar B$. When is $\bar\sigma = 0$, and what is U then?

(c) Compute $\mathbb{E}[U_T]$, and find the condition on μ_1, μ_2 under which U is a martingale. Is $\mu_1 = \mu_2$ enough? Interpret for an investor who measures her wealth in shares of stock 2.

Exercise 8.5 *(Two Brownian motions that are not jointly Gaussian)* [⋆⋆] This exercise continues Exercise 7.5. Let $\mathrm{sgn}(x) = 1$ for $x > 0$ and -1 for $x \le 0$, let $\rho_s = 1$ for $s \in [0,1]$ and $\rho_s = \mathrm{sgn}(W_1)$ for $s > 1$, and $B_t = \int_0^t \rho_s\,\mathrm{d}W_s$.

(a) Show that ρ is progressively measurable and that B is a Brownian motion. Write B_t explicitly for $t \ge 1$.

(b) Compute $[W,B]_t$ and $\mathbb{E}[W_t B_t]$.

(c) Show that $\mathbb{P}(W_2 + B_2 > 0) = \frac{3}{8}$, and deduce that (W_2, B_2) is not a Gaussian vector, although W_2 and B_2 are both $\mathcal{N}(0,2)$. *Hint:* write W_1 and $W_2 - W_1$ in terms of two independent $\mathcal{N}(0,1)$ variables, and use the rotational invariance of the standard Gaussian law on $\mathbb{R}^2$.

(d) The extreme case. Let W and B be Brownian motions for the same filtration with $[W,B]_t = t$ for every t. Show that $B_t = W_t$ for all t, almost surely. *Hint:* compute $\mathbb{E}[V_2(W-B;\Pi)]$ for a partition Π of $[0,t]$ in two ways, and use $\mathbb{E}[V_2(W;\Pi)^2] = t^2 + 2\sum_j h_j^2$ from the proof of Theorem 5.16.

(e) What do (a)–(d) say about how the correlation of two assets should be specified in a model?

Exercise 8.6 *(The squared norm of a Brownian motion)* [⋆⋆] Let $W = (W^1,\dots,W^d)$ be a d-dimensional Brownian motion, $d \ge 1$, and $R_t = |W_t|^2 = \sum_k (W^k_t)^2$. Let $e(x) = x/|x|$ for $x \in \mathbb{R}^d \setminus \{0\}$ and $e(0) = (1,0,\dots,0)$.

(a) Show that $\mathrm{d}R_t = d\,\mathrm{d}t + 2\sum_k W^k_t\,\mathrm{d}W^k_t$, and deduce $\mathbb{E}[R_t]$.

(b) Show that $\beta_t = \sum_k \int_0^t e^k(W_s)\,\mathrm{d}W^k_s$ is a Brownian motion, and that $R_t = d\,t + 2\int_0^t \sqrt{R_s}\,\mathrm{d}\beta_s$, with the integral understood as in Theorem 8.10.

(c) Compute $\mathbb{E}[R_t^2]$ with Itô's formula, and check your answer against the law of R_t/t.

(d) For $d = 1$, show that β coincides almost surely with $\int_0^t \operatorname{sgn}(W_s)\,\mathrm{d}W_s$ of Exercise 8.5, although e and sgn differ at 0.

Exercise 8.7 *(Itô and Stratonovich sums)* [⋆⋆] This is a computational exercise, with a theoretical first part. For $f \in C^2(\mathbb{R})$ and a partition Π of $[0,t]$, let

$$I_\Pi = \sum_j f'(W_{t_j})\,\delta_j W, \qquad J_\Pi = \sum_j \tfrac{1}{2}\bigl(f'(W_{t_j}) + f'(W_{t_{j+1}})\bigr)\,\delta_j W.$$

(a) Show that $J_\Pi \to f(W_t) - f(0)$ in probability as $\|\Pi\| \to 0$: trapezoid sums obey the chain rule of ordinary calculus. This generalises the case $\lambda = \frac{1}{2}$ of Theorem 7.11(ii).

(b) Take $f(x) = \cos x$, $t = 1$ and uniform partitions with $h = 2^{-k}$, $k = 2, \dots, 14$. Simulate at least 1000 Brownian paths on the finest grid, and compute for each h the root-mean-square error, as an approximation of $f(W_1) - f(0)$, of (i) I_Π, (ii) $I_\Pi + \frac{1}{2}\sum_j f''(W_{t_j})\,h$, and (iii) J_Π. Plot the three errors against h on logarithmic axes.

(c) Which errors tend to 0, and at which rates? For (ii), show that the error is

$$\frac{1}{2}\sum_j f''(W_{t_j})\bigl((\delta_j W)^2 - h\bigr)$$

plus terms of higher order, compute the mean square of this leading term to first order in h, and compare with your simulation. For (i), identify the limit.

(d) The sums J_Π converge faster and obey the ordinary chain rule. Why are gains from trading still modelled with the sums I_Π?

Exercise 8.8 *(Tanaka's formula)* [⋆⋆⋆] Itô's formula does not apply to $f(x) = |x|$, which is not differentiable at 0. Let sgn be as in Exercise 8.5, $B_t = \int_0^t \operatorname{sgn}(W_s)\,\mathrm{d}W_s$, and $f_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$ for $\varepsilon > 0$.

(a) What would Itô's formula, applied to $|x|$ without justification, predict for $|W_t|$? Why is the prediction false?

(b) Write Itô's formula for $f_\varepsilon(W_t)$.

(c) Show that $\mathbb{E}\int_0^t \mathbf{1}\{W_s = 0\}\,\mathrm{d}s = 0$, and that $\int_0^t f_\varepsilon'(W_s)\,\mathrm{d}W_s \to B_t$ in L^2 as $\varepsilon \to 0$.

(d) Deduce that $L_t^\varepsilon = \frac{1}{2}\int_0^t f_\varepsilon''(W_s)\,\mathrm{d}s$ converges in L^2 to $L_t = |W_t| - B_t$ for every t, so that $|W_t| = B_t + L_t$ (Tanaka's formula). Show that L has continuous paths, $L_0 = 0$, L is nondecreasing, and $\mathbb{E}[L_t] = \sqrt{2t/\pi}$.

(e) Let $0 \le a < b$ and $\delta > 0$. Show that $L_b = L_a$ almost surely on the event $\{|W_s| \ge \delta \text{ for all } s \in [a,b]\}$: L increases only when W is at 0. Compare with the discrete decomposition of $|S_n|$ in Exercise 2.1(d).

Chapter 9

Stochastic differential equations

Exercise 9.1 *(Checking the hypotheses)* [⋆] For each equation, decide whether Theorem 9.5 applies on every interval $[0,T]$, and if not, which hypothesis fails.

(a) $\mathrm{d}X_t = \tfrac{1}{2}X_t\,\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}W_t$, $X_0 = \sinh(a)$ with $a \in \mathbb{R}$. Show that $X_t = \sinh(a + W_t)$ is a solution. Is it the only one?

(b) $\mathrm{d}X_t = |X_t|\,\mathrm{d}W_t$, $X_0 = x_0$. Find the solution for $x_0 > 0$, $x_0 = 0$ and $x_0 < 0$. Can a path change sign?

(c) $\mathrm{d}X_t = -X_t^3\,\mathrm{d}t + \mathrm{d}W_t$. Show that the ordinary differential equation $x' = -x^3$, $x(0) = x_0$, has a solution on $[0,\infty)$ for every x_0, while $x' = x^3$ has none when $x_0 \neq 0$. What does this say about the growth condition of the theorem?

(d) $\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t^+}\,\mathrm{d}W_t$ with $\kappa, \theta, \sigma > 0$ (the Cox–Ingersoll–Ross short-rate model). Which result quoted in the notes still gives pathwise uniqueness?

Exercise 9.2 *(Ornstein–Uhlenbeck correlations)* [⋆] Let X be the Ornstein–Uhlenbeck process of Theorem 9.8, $\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$ with $\kappa, \sigma > 0$. You may use that $\mathbb{E}\big[\int_0^t f\,\mathrm{d}W \int_0^t g\,\mathrm{d}W\big] = \int_0^t f(s)g(s)\,\mathrm{d}s$ for deterministic $f, g \in L^2[0,t]$ (polarisation of the Itô isometry).

(a) For $X_0 = x_0$ constant and $0 \le s \le t$, show that $\operatorname{Cov}(X_s, X_t) = \frac{\sigma^2}{2\kappa}\big(e^{-\kappa(t-s)} - e^{-\kappa(t+s)}\big)$.

(b) Now let $X_0 \sim \mathcal{N}(\theta, \sigma^2/(2\kappa))$ be $\mathcal{F}_0$-measurable. Show that $\operatorname{Cov}(X_s, X_t) = \frac{\sigma^2}{2\kappa}e^{-\kappa|t-s|}$ for all $s, t \ge 0$. After how long is the correlation between two observations one half?

(c) With X_0 as in (b), fix $\Delta > 0$ and let $Y_k = X_{k\Delta}$. Show that $Y_{k+1} = \theta + \phi(Y_k - \theta) + \varepsilon_{k+1}$ with $\phi = e^{-\kappa\Delta}$ and $\varepsilon_1, \varepsilon_2, \dots$ independent $\mathcal{N}(0, v)$ variables, ε_{k+1} independent of $Y_0, \dots, Y_k$: sampled at regular times, the process is a Gaussian autoregression of order one. Give v in terms of κ, σ, Δ. How would you estimate κ from daily observations of a short rate?

Exercise 9.3 *(The Brownian bridge as a diffusion)* [⋆⋆] Let $a, c \in \mathbb{R}$ and consider $\mathrm{d}X_t = \frac{c - X_t}{1-t}\,\mathrm{d}t + \mathrm{d}W_t$, $X_0 = a$, for $t \in [0,1)$.

(a) Show that Theorem 9.5 applies on $[0,T]$ for every $T < 1$, but not on $[0,1]$. Explain the

drift: where does it push the path, and how hard, as $t \uparrow 1$?

(b) Show that $X_t = a(1-t) + ct + (1-t)\int_0^t \frac{\mathrm{d}W_s}{1-s}$ for $t < 1$.

(c) Show that X is a Gaussian process with $\mathbb{E}[X_t] = a(1-t) + ct$ and $\mathrm{Cov}(X_s, X_t) = s(1-t)$ for $0 \le s \le t < 1$. Deduce that $X_t \to c$ in L^2 as $t \uparrow 1$.

(d) Take $a = 0$ and let $Z_t = W_t - tW_1 + tc$ for $t \in [0,1]$. Show that $(Z_t)_{t<1}$ and $(X_t)_{t<1}$ have the same finite-dimensional laws, and that $(W_t - tW_1)_{t\le 1}$ is independent of W_1. In what sense is X a Brownian motion conditioned to end at c? Compare with Exercise 5.3. What is the advantage of X over Z as a description of this process?

Exercise 9.4 *(Moments of linear equations)* [⋆⋆] Let $\mu_0, \mu_1, \sigma_0, \sigma_1$ be constants, $x \in \mathbb{R}$, and let X solve $\mathrm{d}X_t = (\mu_0 + \mu_1 X_t)\,\mathrm{d}t + (\sigma_0 + \sigma_1 X_t)\,\mathrm{d}W_t$, $X_0 = x$. You may use that $\mathbb{E}[\sup_{t\le T} X_t^4] < \infty$ for every T. Let $m(t) = \mathbb{E}[X_t]$ and $q(t) = \mathbb{E}[X_t^2]$.

(a) Using Theorem 9.11(a) with $f(x) = x$, show that $m' = \mu_0 + \mu_1 m$, and solve this equation.

(b) Using $f(x) = x^2$ (justify that M^f is a martingale), show that $q' = (2\mu_1 + \sigma_1^2)\,q + 2(\mu_0 + \sigma_0\sigma_1)\,m + \sigma_0^2$.

(c) Suppose $\mu_1 < 0$ and $2\mu_1 + \sigma_1^2 < 0$. Find the limits of $m(t)$, $q(t)$ and $\mathrm{Var}(X_t)$ as $t \to \infty$, and check your answer against Theorem 9.8 for the Ornstein–Uhlenbeck process.

(d) Take $\mu_0 = 1$, $\mu_1 = -1$, $\sigma_0 = 0$, $\sigma_1 = 2$ and $x = 1$. Show that $\mathbb{E}[X_t] = 1$ for all t while $\mathrm{Var}(X_t) \to \infty$. The drift pulls X towards 1; what goes wrong?

Exercise 9.5 *(Stationary densities)* [⋆⋆] Let $I = (l, r)$ be an open interval ($l = -\infty$ and $r = \infty$ are allowed), let $b \in C^1(I)$ and $\sigma \in C^2(I)$ with $\sigma > 0$ on I, and consider solutions of $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$ that stay in I. A probability density p on I is *stationary* if, whenever X_0 is $\mathcal{F}_0$-measurable with density p, X_t has density p for every $t \ge 0$. Write $\mathcal{L}$ for the generator.

(a) Show that if p is stationary, then $\int_I (\mathcal{L}f)(x)\,p(x)\,\mathrm{d}x = 0$ for every $f \in C^2(\mathbb{R})$ whose support is a compact subset of I.

(b) Let $p \in C^2(I)$. Show that the condition in (a) holds if and only if $\frac{1}{2}(\sigma^2 p)'' - (bp)' = 0$ on I. Show that

$$p(x) = \frac{C}{\sigma(x)^2} \exp\Big(\int_{x_0}^{x} \frac{2b(u)}{\sigma(u)^2}\,\mathrm{d}u \Big),$$

with $x_0 \in I$ fixed, satisfies this equation, and is a probability density for a suitable constant C whenever the function of x on the right is integrable on I. So (a) only singles out a candidate; in (c)–(e) you may take for granted that this candidate is indeed stationary (the converse needs arguments beyond the notes).

(c) Apply (b) to the Ornstein–Uhlenbeck process and compare with Theorem 9.8.

(d) Apply (b) to the Cox–Ingersoll–Ross process $\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t$ on $I = (0,\infty)$, with $\kappa, \theta, \sigma > 0$ and $2\kappa\theta \ge \sigma^2$ (under this Feller condition, which you may take for granted, solutions started in $(0,\infty)$ stay there). Identify the law and its mean.

(e) Apply (b) to $\mathrm{d}X_t = (1 - X_t)\,\mathrm{d}t + 2X_t\,\mathrm{d}W_t$ on $I = (0,\infty)$, after checking with Theorem 9.7 that the solution from $X_0 > 0$ stays positive. Show that the density has mean 1 and infinite

variance, and compare with Exercise 9.4(d).

Exercise 9.6 *(Feller's diffusion reaches zero)* [⋆⋆] Let $\sigma, s > 0$ and let S solve $\mathrm{d}S_t = \sigma\sqrt{S_t}\,\mathrm{d}W_t$, $S_0 = s$, with $S_t \geq 0$ for all t: the CEV model with $\gamma = \frac{1}{2}$ and $\mu = 0$. You may assume that such a solution exists. Fix $T > 0$ and $\lambda > 0$.

(a) Look for $M_t = \exp(-\ell(t)S_t)$ with $\ell : [0,T] \to (0,\infty)$ continuously differentiable and $\ell(T) = \lambda$. Show that M has no $\mathrm{d}t$ term if $\ell' = \frac{\sigma^2}{2}\ell^2$, and solve this equation.

(b) Show that M is a martingale on $[0,T]$, and deduce that

$$\mathbb{E}[e^{-\lambda S_T}] = \exp\Big(-\frac{\lambda s}{1+\lambda\sigma^2 T/2}\Big).$$

(c) Deduce that $\mathbb{P}(S_T = 0) = \exp\big(-2s/(\sigma^2 T)\big)$. In the setting of Figure 9.2 ($s = 1$, $\sigma = 0.6$, $\gamma = \frac{1}{2}$), what fraction of the paths should be at 0 at time $T = 5$? Compare with the right panel of the figure.

(d) Compute $\mathbb{E}[S_T]$ and $\mathrm{Var}(S_T)$ from (b).

Exercise 9.7 *(A CEV price that is not a martingale)* [⋆⋆⋆] Let $B = (B^1, B^2, B^3)$ be three independent Brownian motions, $x = (1,0,0)$, $Y_t = x + B_t$, $R_t = |Y_t|$ and $S_t = 1/R_t$. You may use without proof: (i) almost surely $Y_t \neq 0$ for all $t \geq 0$, and the multidimensional Itô formula (Theorem 8.17) applies to $f(Y_t)$ for $f \in C^2(\mathbb{R}^3 \setminus \{0\})$; (ii) if $Z \sim \mathcal{N}(0, I_3)$, then $|Z|$ and $Z/|Z|$ are independent, $Z/|Z|$ is uniform on the unit sphere, and $|Z|$ has density $\sqrt{2/\pi}\,r^2 e^{-r^2/2}$ on $(0,\infty)$; (iii) Newton's theorem: if U is uniform on the unit sphere and $\rho > 0$, then $\mathbb{E}[1/|x+\rho U|] = 1/\max(|x|,\rho)$.

(a) Show that $S_t = 1 - \sum_{i=1}^3 \int_0^t \frac{Y_u^i}{R_u^3}\,\mathrm{d}B_u^i$. Deduce that S is a local martingale.

(b) Show that $\beta_t = -\sum_i \int_0^t \frac{Y_u^i}{R_u}\,\mathrm{d}B_u^i$ is a Brownian motion. Using that, for $H \in \mathbb{H}^2_{\mathrm{loc}}$,

$$\int_0^t H_u\,\mathrm{d}\beta_u = -\sum_{i=1}^3 \int_0^t H_u \frac{Y_u^i}{R_u}\,\mathrm{d}B_u^i,$$

show that $\mathrm{d}S_t = S_t^2\,\mathrm{d}\beta_t$: S is a CEV price with $\mu = 0$, $\sigma = 1$ and $\gamma = 2$.

(c) Show that $\mathbb{E}[S_t] = \mathbb{E}[1/\max(1, \sqrt{t}|Z|)]$ with $Z \sim \mathcal{N}(0, I_3)$, and then that $\mathbb{E}[S_t] = 2N(1/\sqrt{t}) - 1$. Deduce that $\mathbb{E}[S_t] < S_0$ for every $t > 0$ and $\mathbb{E}[S_t] \to 0$ as $t \to \infty$.

(d) Conclude that S is a positive local martingale that is not a martingale. Show that $\mathbb{E}[S_u^4] = \infty$ for every $u > 0$, so that the integrand S^2 in $\mathrm{d}S = S^2\,\mathrm{d}\beta$ is not in $\mathbb{H}^2$, and say which hypothesis of Theorem 9.5 fails.

(e) Suppose that S is the price of a stock in a market with zero interest rate, and that every claim paying H at time T is priced at $\mathbb{E}[H]$. Compare the price of the claim paying S_T at time T with the price S_0 of the stock, which delivers the same amount at T. Comment.

Exercise 9.8 *(The Milstein scheme)* [⋆⋆⋆] This exercise is partly computational. Let S be geometric Brownian motion, $\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$, and S^h its Euler–Maruyama approximation with step $h = T/n$ (Theorem 9.17), driven by the same Brownian path.

(a) Show that $\mathbb{E}[(S_T^h)^2] = S_0^2\big((1+\mu h)^2+\sigma^2 h\big)^n$ and $\mathbb{E}[S_T S_T^h] = S_0^2\big(e^{\mu h}(1+\mu h+\sigma^2 h)\big)^n$. Deduce that

$$\mathbb{E}\big[(S_T - S_T^h)^2\big] = \tfrac{1}{2}\sigma^4 S_0^2 e^{(2\mu+\sigma^2)T}\, T\, h + O(h^2).$$

So the root-mean-square error is of order exactly $h^{1/2}$, as in Theorem 9.16(a).

(b) On $[t_k, t_{k+1}]$, the Euler scheme freezes the diffusion σS_u at σS_{t_k}. Freezing it instead at its first-order expansion $\sigma S_{t_k}\big(1 + \sigma(W_u - W_{t_k})\big)$, and using Theorem 7.11 for $\int_{t_k}^{t_{k+1}} (W_u - W_{t_k})\,\mathrm{d}W_u$, derive the *Milstein scheme*

$$S^{\mathrm{M}}_{k+1} = S^{\mathrm{M}}_k\Big(1 + \mu h + \sigma\Delta W_k + \tfrac{1}{2}\sigma^2\big((\Delta W_k)^2 - h\big)\Big), \qquad \Delta W_k = W_{t_{k+1}} - W_{t_k}.$$

(c) Take $S_0 = 1$, $\mu = 0.05$, $\sigma = 0.4$, $T = 1$. Simulate 10^5 Brownian paths on a grid of 2^{10} steps, and use them (summing increments) for $h = 2^{-2}, \dots, 2^{-10}$. Estimate the root-mean-square errors $\mathbb{E}[(S_T - S_T^h)^2]^{1/2}$ and $\mathbb{E}[(S_T - S_T^{\mathrm{M}})^2]^{1/2}$ against the exact solution on the same paths, compare the first with (a), and estimate both orders of convergence by a least-squares fit on log–log axes.

Part IV

Pricing and hedging

Chapter 10

The Black–Scholes equation by hedging

Exercise 10.1 *(Constant proportions)* [⋆] An investor keeps a fixed fraction $\pi \in \mathbb{R}$ of her wealth in the stock and the rest in the money market. Let $X_0 > 0$ and

$$X_t = X_0 \exp\Big(\Big(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}\Big)t + \pi\sigma W_t\Big), \qquad \Delta_t = \frac{\pi X_t}{S_t}.$$

(a) Show that (X_0, Δ) is a self-financing strategy whose value is X. *Hint:* Theorem 10.2.

(b) Compute $\mathbb{E}[X_T]$, and show that $\frac{1}{t}\log X_t \to r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}$ almost surely as $t \to \infty$ (you may use that $W_t/t \to 0$ almost surely).

(c) Find the fraction π^* that maximises this growth rate. With $\mu = 8\%$, $r = 3\%$ and $\sigma = 20\%$, compute π^* and the growth rates of π^*, of $\pi = 1$ (all in the stock) and of $\pi = 2\pi^*$. Which π maximises $\mathbb{E}[X_T]$?

(d) Is the value of this strategy bounded below? Compare with Theorem 10.3.

Exercise 10.2 *(Hedging a book of options)* [⋆] A desk has sold 10,000 calls with strike 105 and bought 5,000 puts with strike 95, all expiring in three months, on a stock at $S = 100$, with $r = 3\%$ and $\sigma = 25\%$. You may use `call`, `put` and `greeks` from `code/src/ieor222/blackscholes.py`.

(a) Explain why the delta of the book is the sum of the deltas of its options, and how many shares the desk must hold to be delta-neutral.

(b) Compute the gamma of the book. Using the gamma–theta decomposition Equation (10.12), say whether the delta-hedged book gains or loses, over one day, if the stock moves by more than $\sigma\sqrt{h}$ in either direction.

(c) The desk can also trade calls with strike 100 expiring in six months. How many must it buy to make the book gamma-neutral, and how many shares must it then hold to be delta-neutral?

(d) The stock jumps instantly to 103, and then, in another scenario, to 97. Compute the change in value of the delta-hedged book of (a) and of the delta- and gamma-hedged book of (c), and explain the difference.

Exercise 10.3 *(Self-financing in terms of holdings)* [⋆⋆] Let $B_t = e^{rt}$, and let Δ and η be Itô processes: the numbers of shares and of units of the money market held at time t. The value of the portfolio is $X_t = \Delta_t S_t + \eta_t B_t$.

(a) Using Theorem 8.18, show that X satisfies the wealth equation Equation (10.6) if and only if

$$S_t \, \mathrm{d}\Delta_t + B_t \, \mathrm{d}\eta_t + \mathrm{d}[\Delta, S]_t = 0.$$

(b) In discrete time, with rebalancing at dates t_k, self-financing means $S_{t_{k+1}}(\Delta_{t_{k+1}} - \Delta_{t_k}) + B_{t_{k+1}}(\eta_{t_{k+1}} - \eta_{t_k}) = 0$: the new positions are bought at the new prices with the value of the old ones. Explain how the covariation term in (a) arises from this condition.

(c) For the claim paying S_T^2 (Theorem 10.10), write Δ and η of the delta hedge explicitly, and check the condition of (a) directly.

Exercise 10.4 *(Time-dependent rate and volatility)* [⋆⋆] Let $r(t)$ and $\sigma(t) > 0$ be continuous functions on $[0, T]$, let the money market be $B_t = \exp(\int_0^t r(s)\,\mathrm{d}s)$ and the stock $S_t = S_0 \exp\big(\int_0^t \sigma(s)\,\mathrm{d}W_s + \int_0^t (\mu - \tfrac{1}{2}\sigma(s)^2)\,\mathrm{d}s\big)$. A self-financing strategy is defined as in Theorem 10.1, with B_t in place of e^{rt}: $X_t/B_t = X_0 + \int_0^t \Delta_u \, \mathrm{d}(S_u/B_u)$.

(a) Show that $\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma(t) S_t \, \mathrm{d}W_t$, compute $\mathrm{d}(S_t/B_t)$, and show that S_u has a density that is positive on $(0, \infty)$ for $u > 0$.

(b) Let g be Borel and suppose that a self-financing strategy replicates $g(S_T)$ with value $X_t = v(t, S_t)$, where $v \in C^{1,2}([0, T) \times (0, \infty))$. Show that

$$\partial_t v + r(t)\, x\, \partial_x v + \frac{\sigma(t)^2 x^2}{2} \, \partial_{xx} v = r(t)\, v \quad \text{on } [0, T) \times (0, \infty),$$

and that $\Delta_t = \partial_x v(t, S_t)$. You may follow the proof of Theorem 10.5.

(c) Let $\rho(t) = \int_t^T r(s)\,\mathrm{d}s$ and $\Sigma(t)^2 = \int_t^T \sigma(s)^2\,\mathrm{d}s$. Show that $v(t, x) = xN(d_+) - Ke^{-\rho(t)}N(d_-)$, with $d_\pm = \big(\log(x/K) + \rho(t) \pm \tfrac{1}{2}\Sigma(t)^2\big)/\Sigma(t)$, solves the equation of (b). So the call is priced by the Black–Scholes formula with the average rate and the root-mean-square volatility over $[t, T]$. *Hint:* write v as a function of x, ρ and Σ^2 and compute its derivatives from Theorem 6.14.

(d) A stock has volatility 10% for six months and 30% for the next six. Compare the one-year at-the-money call priced with the root-mean-square volatility and with the average volatility, for $S_0 = K = 100$ and $r = 0$.

Exercise 10.5 *(Discrete hedging of a digital option)* [⋆⋆] This is a computational exercise. Take $S_0 = K = 100$, $T = 1$, $r = 5\%$, $\sigma = 20\%$ and $\mu = 10\%$. The digital call pays $\mathbf{1}_{\{S_T > K\}}$; its price is $D(t, x) = e^{-r\tau}N(d_-)$ (Theorem 6.13) and its delta $\partial_x D = e^{-r\tau}N'(d_-)/(\sigma x\sqrt{\tau})$ (Exercise 6.5).

(a) Which hypotheses of Theorem 10.7 and of (H) in Section 10.4 fail for the digital?

(b) Simulate the discrete delta hedge of the digital with $n = 4, 8, \dots, 4096$ rebalancing dates and at least 10^4 paths, and estimate the root-mean-square of the terminal error. Do the same for the call with strike K, on the same paths. Estimate the rate β in rms $\approx Cn^{-\beta}$ for both, by regression on a log-log scale.

(c) Look at the paths with the largest errors for $n = 256$. Where does S end, and when is the error made? Explain heuristically why the rate is slower than for the call.

Exercise 10.6 *(The squared claim, exactly)* [⋆⋆] Let $g(x) = x^2$, let $\mu = r$, and use the notation of Section 10.4. Write $U_k = 1 + R_k$ and $Y_t = e^{-rt} S_t$.

(a) Show that the profit P_k of Equation (10.15) equals $c_k Y_{t_k}^2 D_k$, with $c_k = e^{rT + \sigma^2 (T - t_k)}$ and $D_k = e^{-\sigma^2 h} U_k^2 - 2U_k + 1$.

(b) Show that $\mathbb{E}[D_k] = 0$ and $\mathbb{E}[D_k^2] = e^{4\sigma^2 h} - 4e^{2\sigma^2 h} + 4e^{\sigma^2 h} - 1$.

(c) Deduce that

$$\mathbb{E}[e_n^2] = \left(e^{4\sigma^2 h} - 4e^{2\sigma^2 h} + 4e^{\sigma^2 h} - 1\right) S_0^4 \, e^{2(r+\sigma^2)T} \sum_{k=0}^{n-1} e^{4\sigma^2 t_k},$$

and check that it agrees with Equation (10.13) to first order in h.

(d) For $S_0 = 100$, $T = 1$, $r = 5\%$ and $\sigma = 40\%$, compare the root-mean-square of e_n with the price of the claim for $n = 12$, 52 and 252.

Exercise 10.7 *(Hedging with the wrong volatility)* [⋆⋆⋆] A dealer sells a call with strike K and maturity T at the Black–Scholes price computed with a volatility $\hat{\sigma} > 0$, and hedges it continuously with $\Delta_t = \partial_x \hat{c}(t, S_t)$, where $\hat{c}$ is the call function of Theorem 6.11 with $\hat{\sigma}$ in place of σ. The stock, however, satisfies

$$S_t = S_0 \exp\Big(\int_0^t \sigma_u \, \mathrm{d}W_u + \int_0^t \Big(\mu_u - \frac{\sigma_u^2}{2}\Big) \mathrm{d}u \Big),$$

where μ and σ are bounded progressively measurable processes. Define the value of the strategy by $e^{-rt} X_t = \hat{c}(0, S_0) + \int_0^t \Delta_u \, \mathrm{d}(e^{-ru} S_u)$, with $\mathrm{d}(e^{-ru} S_u) = (\mu_u - r) e^{-ru} S_u \, \mathrm{d}u + \sigma_u e^{-ru} S_u \, \mathrm{d}W_u$, and write $\hat{\Gamma} = \partial_{xx} \hat{c}$.

(a) Show that, almost surely, for every $t < T$,

$$e^{-rt}(X_t - \hat{c}(t, S_t)) = \frac{1}{2} \int_0^t e^{-ru} \, \hat{\Gamma}(u, S_u) \, S_u^2 \, (\hat{\sigma}^2 - \sigma_u^2) \, \mathrm{d}u.$$

Hint: fix $t' < T$ and apply Itô's formula (Theorem 8.13) on $[0, t']$ to $f(t, \log S_t)$ with $f(t, z) = e^{-rt} \hat{c}(t, e^z)$, extended beyond t' as in the setting of Chapter 10; then use the equation solved by $\hat{c}$.

(b) Show that $\hat{\Gamma}(u, x) \, x^2 \le x / (\hat{\sigma} \sqrt{2\pi (T - u)})$, and deduce that the identity of (a) holds at $t = T$, with $X_T - (S_T - K)^+$ on the left.

(c) Conclude that if $\sigma_u^2 \le \hat{\sigma}^2$ for all u, the dealer ends with at least the payoff, whatever the path, and that if $\sigma_u^2 \ge \hat{\sigma}^2$ for all u she ends with at most the payoff. Why is the profit weighted by the gamma?

(d) Let $\sigma_u = \sigma$ be constant and $\mu_u = r$. Show that the expected discounted profit is $\hat{c}(0, S_0) - c(0, S_0)$, the difference of the Black–Scholes prices with the two volatilities. Evaluate it for $S_0 = K = 100$, $T = 1$, $r = 5\%$, $\hat{\sigma} = 20\%$, $\sigma = 25\%$, and compare with the vega approximation.

Chapter 11

Change of measure and risk-neutral pricing

Exercise 11.1 *(Two ways to compute under $\mathbb{Q}$)* [$\star$] Let $\theta \in \mathbb{R}$, let $\mathbb{Q}$ have density $Z_T = \exp(-\theta W_T - \frac{\theta^2}{2}T)$ with respect to $\mathbb{P}$, and let $\widetilde{W}_t = W_t + \theta t$, as in Theorem 11.2. Let $0 \le t \le T$.

(a) Using Theorem 11.2, compute $\mathbb{E}^{\mathbb{Q}}[W_T \mid \mathcal{F}_t]$, the variance of W_T under $\mathbb{Q}$, and $\mathbb{Q}(W_T > 0)$.

(b) Compute $\mathbb{E}^{\mathbb{Q}}[W_T \mid \mathcal{F}_t]$ again, this time with expectations under $\mathbb{P}$ only, by Bayes' formula (Theorem 11.1) and the independence lemma (Theorem 1.19).

(c) Show that $1/Z_t = \exp(\theta\widetilde{W}_t - \frac{\theta^2}{2}t)$, that $(1/Z_t)_{t\le T}$ is a $\mathbb{Q}$-martingale, and that $\mathbb{P}(A) = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A/Z_T]$ for every event A. Explain why this makes the roles of $\mathbb{P}$ and $\mathbb{Q}$ symmetric: what does Girsanov's theorem, applied under $\mathbb{Q}$ to $\widetilde{W}$ with the constant $-\theta$, say?

Exercise 11.2 *(Time-dependent coefficients)* [$\star\star$] Let $r, \mu, \sigma : [0,T] \to \mathbb{R}$ be continuous functions with $\sigma \ge \varepsilon > 0$: the market of Section 11.3 with deterministic coefficients, money market $B_t = \exp\big(\int_0^t r(u)\,\mathrm{d}u\big)$ and stock $S_t = S_0 \exp\big(\int_0^t \sigma(u)\,\mathrm{d}W_u + \int_0^t (\mu(u) - \tfrac{1}{2}\sigma(u)^2)\,\mathrm{d}u\big)$. Let $(\mathcal{F}_t)$ be the augmented filtration of W, let $\mathbb{Q}$ and $\widetilde{W}$ be as in Theorem 11.7, and for $0 \le t < T$ and $\tau = T - t$ let

$$\bar{r}(t) = \frac{1}{\tau}\int_t^T r(u)\,\mathrm{d}u, \qquad \bar{\sigma}(t)^2 = \frac{1}{\tau}\int_t^T \sigma(u)^2\,\mathrm{d}u.$$

(a) Show that under $\mathbb{Q}$ the log-return $\log(S_T/S_t)$ is normal and independent of $\mathcal{F}_t$, and give its mean and variance. You may use Theorem 7.10.

(b) Show that the price at time t of the call with strike $K > 0$ is the Black–Scholes price of Theorem 6.11 with the constants r and σ replaced by $\bar{r}(t)$ and $\bar{\sigma}(t)$.

(c) Let C_t be this price and $D_t = 1/B_t$. For $t < T$, write D_tC_t as an Itô process driven by $\widetilde{W}$, and use Theorem 8.11 and Equation (11.6) to show that the hedge of Theorem 11.11(iii) is $\Delta_t = N(d_+)$, with d_+ computed from $\bar{r}(t)$, $\bar{\sigma}(t)$ and S_t. You may use the delta of Theorem 6.14.

(d) A trader says that only the total variance $\int_t^T \sigma(u)^2\,\mathrm{d}u$ until the payment date T matters.

Is she right for calls? Take r constant and $\sigma(u)^2 = \sigma_0^2(1 + \kappa \sin(2\pi u/T))$ with $\sigma_0 > 0$ and $|\kappa| < 1$, and consider the claim paying $(S_{T/2} - K)^+$ at time T. Show that its price at time 0 depends on κ, although $\int_0^T \sigma(u)^2 \, du$ does not.

Exercise 11.3 *(One source of risk, two stocks)* [⋆⋆] Let $(\mathcal{F}_t)$ be the augmented filtration of W, let r be a constant rate, and let two stocks follow $dS_t^i = \mu_i S_t^i \, dt + \sigma_i S_t^i \, dW_t$, with $S_0^i > 0$, $\sigma_i > 0$ and market prices of risk $\theta_i = (\mu_i - r)/\sigma_i$, $i = 1, 2$. A self-financing strategy holds Δ_t^1 and Δ_t^2 shares and has wealth $dX_t = \Delta_t^1 \, dS_t^1 + \Delta_t^2 \, dS_t^2 + r(X_t - \Delta_t^1 S_t^1 - \Delta_t^2 S_t^2)dt$.

(a) Show that there is a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ under which $e^{-rt}S_t^1$ and $e^{-rt}S_t^2$ are both martingales if and only if $\theta_1 = \theta_2$. *Hint:* Theorem 11.11(ii) for each stock alone.

(b) Let $\theta_1 > \theta_2$. Compute the discounted wealth of the strategy with $X_0 = 0$, $\Delta_t^1 = 1/(\sigma_1 S_t^1)$ and $\Delta_t^2 = -1/(\sigma_2 S_t^2)$, and show that it is a tame arbitrage. Describe the position in words.

(c) Let $\theta_1 = \theta_2 = \theta$ and let $\mathbb{Q}$ be the common risk-neutral measure. Let $V > 0$ be the price of a third traded asset, an Itô process with $dV_t = \alpha_t V_t \, dt + \beta_t V_t \, dW_t$, whose discounted price is a $\mathbb{Q}$-martingale. Show that $\alpha_t - r = \theta \beta_t$, $dt \otimes d\mathbb{P}$-almost everywhere: every traded asset driven by W earns the same excess return per unit of volatility. (For a call, this is the elasticity statement of Theorem 10.6(b).)

Exercise 11.4 *(Dividends, Black's formula and early exercise)* [⋆] Work in the setting of Theorem 11.14, with $K > 0$, $0 \le t < T$ and $\tau = T - t$, and let $F_t = S_t e^{(r-q)\tau}$ be the forward price.

(a) Show that $C_t = e^{-r\tau}\big(F_t N(d_+) - K N(d_-)\big)$ with $d_\pm = \big(\log(F_t/K) \pm \tfrac{1}{2}\sigma^2\tau\big)/(\sigma\sqrt{\tau})$ (Black's formula), and that $C_t - P_t = e^{-r\tau}(F_t - K)$. Why is it natural that the price depends on S_t, r and q only through F_t and the discount factor?

(b) Show that C_t decreases and P_t increases when q increases, the other parameters being fixed.

(c) Let $q > 0$. Using put–call parity and a bound on P_t, show that $C_t - (S_t - K) \to -\infty$ as $S_t \to \infty$. Compute C_t and $S_t - K$ for $S_t = 150$, $K = 100$, $\tau = 1$, $r = 3\%$, $q = 5\%$ and $\sigma = 20\%$. What does this imply for an American call on this stock? Compare with Theorem 4.13(iv).

Exercise 11.5 *(Exchange rates from both sides)* [⋆⋆] Let X be the price in dollars of one euro, with $dX_t = (r - r_f)X_t \, dt + \sigma X_t \, d\widetilde{W}_t$ under the dollar risk-neutral measure $\mathbb{Q}$ of Theorem 11.14(iii), where r and r_f are the dollar and euro interest rates and $\sigma > 0$. Let $Y_t = 1/X_t$, the price in euros of one dollar.

(a) Compute dY_t. Is Y a geometric Brownian motion with drift $r_f - r$ under $\mathbb{Q}$?

(b) A European investor measures wealth in units of the euro money market. Let $\mathbb{Q}^f$ be the measure of Theorem 11.13 for the numeraire $N_t = X_t e^{r_f t}$, the euro money market account valued in dollars. Show that $W_t^f = \widetilde{W}_t - \sigma t$ is a $\mathbb{Q}^f$-Brownian motion, that $dY_t = (r_f - r)Y_t \, dt - \sigma Y_t \, dW_t^f$, and that $e^{-r_f t}e^{rt}Y_t$, the dollar money market valued in

euros and discounted at the euro rate, is a $\mathbb{Q}^f$-martingale.

(c) *(Siegel's paradox.)* Compute $\mathbb{E}^{\mathbb{Q}}[X_T]$, $\mathbb{E}^{\mathbb{Q}^f}[Y_T]$ and $\mathbb{E}^{\mathbb{Q}}[Y_T]$, and compare them with the forward rate $F = X_0 e^{(r-r_f)T}$ and with $1/F$. Evaluate them for $X_0 = 1.10$, $r = 4\%$, $r_f = 2\%$, $\sigma = 10\%$ and $T = 1$. Why do the two investors agree on the forward rate but not on the expected exchange rate?

(d) Show that the dollar price of a call on one euro with strike K dollars equals $X_0 K$ times the euro price of a put on one dollar with strike $1/K$ euros, both with maturity T.

Exercise 11.6 *(An option to exchange two stocks)* [⋆⋆⋆] Let $(\widetilde{W}^1, \widetilde{W}^2)$ be a two-dimensional Brownian motion under a measure $\mathbb{Q}$, $\rho \in (-1, 1)$, $\widetilde{B}^1 = \widetilde{W}^1$ and $\widetilde{B}^2 = \rho\widetilde{W}^1 + \sqrt{1-\rho^2}\,\widetilde{W}^2$. For $i = 1, 2$ let $S^i_t = S^i_0 \exp\left(\sigma_i \widetilde{B}^i_t + (r - \frac{\sigma_i^2}{2})t\right)$, with $S^i_0 > 0$, $\sigma_i > 0$ and $r \in \mathbb{R}$: two stocks under a risk-neutral measure. An exchange option pays $(S^1_T - S^2_T)^+$ at T. Let $\bar{\sigma}^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$.

(a) Let $\mathbb{Q}^2$ be the measure with numeraire S^2 of Theorem 11.13. Using Theorem 11.4(c), find a two-dimensional $\mathbb{Q}^2$-Brownian motion $(\widehat{W}^1, \widehat{W}^2)$.

(b) Show that $U = S^1/S^2$ satisfies $U_t = U_0 \exp\left(\bar{\sigma}\bar{B}_t - \frac{1}{2}\bar{\sigma}^2 t\right)$ for a $\mathbb{Q}^2$-Brownian motion $\bar{B}$ that you should identify. How could you have predicted that U is a $\mathbb{Q}^2$-martingale?

(c) Deduce that the price at time 0 is $S^1_0 N(d_+) - S^2_0 N(d_-)$, with $d_\pm = \left(\log(S^1_0/S^2_0) \pm \frac{1}{2}\bar{\sigma}^2 T\right)/(\bar{\sigma}\sqrt{T})$.

(d) Write the price as $V(S^1_0, S^2_0)$. Show that $V(x_1, x_2) = x_2\, g(x_1/x_2)$ for a function g, that $\partial_1 V = N(d_+)$ and $\partial_2 V = -N(d_-)$, and that $V = x_1\partial_1 V + x_2\partial_2 V$. Why does r not appear? Suggest a hedge.

(e) What is the limit of the price as $\bar{\sigma} \downarrow 0$?

Exercise 11.7 *(A second source of risk)* [⋆⋆⋆] Let (W^1, W^2) be a two-dimensional Brownian motion under $\mathbb{P}$ and $(\mathcal{F}_t)$ the augmented filtration it generates. The money market has a constant rate r, and the only stock follows $\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W^1_t$, $\sigma > 0$; let $\theta = (\mu - r)/\sigma$. Self-financing strategies in the stock and the money market are as in Section 11.3, with Δ progressively measurable for this $(\mathcal{F}_t)$.

(a) For $\nu \in \mathbb{R}$, let $\mathbb{Q}^\nu$ have density $\exp\left(-\theta W^1_T - \nu W^2_T - \frac{1}{2}(\theta^2 + \nu^2)T\right)$ with respect to $\mathbb{P}$. Show that every $\mathbb{Q}^\nu$ is a risk-neutral measure, and that $W^1_t + \theta t$ and $W^2_t + \nu t$ are independent $\mathbb{Q}^\nu$-Brownian motions.

(b) A claim pays 1 at T if $W^2_T > 0$: a bet on a risk that the stock does not carry. Compute $e^{-rT}\mathbb{E}^{\mathbb{Q}^\nu}[\mathbf{1}_{\{W^2_T > 0\}}]$, and the set of its values as ν ranges over $\mathbb{R}$.

(c) Show that no self-financing strategy with bounded discounted wealth replicates the claim. *Hint:* under each $\mathbb{Q}^\nu$, write the discounted wealth as an integral against $W^1_t + \theta t$, and use that a bounded local martingale is a martingale.

(d) Show that W^2 is not of the form $\int_0^t \Gamma_u\,\mathrm{d}W^1_u$ with $\Gamma \in \mathbb{H}^2_{\mathrm{loc}}$, and say which hypothesis of Theorem 11.10 fails. Compare with the trinomial model of Theorem 4.14.

Exercise 11.8 *(Reweighting with a random market price of risk)* [⋆⋆] This is a computational

exercise. Let $(\mathcal{F}_t)$ be the augmented filtration of W, let r and $\sigma > 0$ be constants, $a \geq 0$, and $\lambda(s) = a \tanh\big(\log(s/S_0)\big)$ for $s > 0$. Under $\mathbb{P}$, the stock is $S_t = e^{Y_t}$, where

$$\mathrm{d}Y_t = \Big(r + \sigma\lambda(e^{Y_t}) - \frac{\sigma^2}{2}\Big)\mathrm{d}t + \sigma\,\mathrm{d}W_t, \qquad Y_0 = \log S_0 :$$

after a rise, the expected return goes up.

(a) Show that this equation has a unique solution (Theorem 9.5), that $\mathrm{d}S_t = \alpha_t S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$ for a bounded process α that you should identify, and that under the risk-neutral measure of Theorem 11.7 the stock is the Black–Scholes stock Equation (6.3). What is the price of a call, and does it depend on a?

(b) Simulate under $\mathbb{P}$ with the Euler scheme $Y_{j+1} = Y_j + (r + \sigma\lambda_j - \frac{\sigma^2}{2})h + \sigma\Delta W_j$, $\lambda_j = \lambda(e^{Y_j})$, on the grid $t_j = jh$, $h = T/n$, and weight each path by $Z^h = \exp\big(-\sum_j \lambda_j \Delta W_j - \frac{h}{2}\sum_j \lambda_j^2\big)$. Show that the weighted average of $e^{-rT}(e^{Y_n} - K)^+$ is an unbiased estimator of the Black–Scholes price for every h, although the Euler scheme does not have the law of Y_T. *Hint:* under the measure with density Z^h, the variables $\Delta W_j + \lambda_j h$ are independent $\mathcal{N}(0, h)$; condition on $\mathcal{F}_{t_{n-1}}$, use Theorem 1.26, and induct.

(c) Implement it with $S_0 = K = 100$, $r = 3\%$, $\sigma = 20\%$, $T = 1$, 10^5 paths, $n = 50$ and $n = 250$, and $a = 1$ and $a = 3$. Report the weighted estimates with 95% confidence intervals, the plain averages of $e^{-rT}(e^{Y_n} - K)^+$, and the effective sample size $(\sum_i Z_i^h)^2 / \sum_i (Z_i^h)^2$. Comment on what the plain averages estimate, and on the size of the confidence intervals for the two values of a.

Chapter 12

The Feynman–Kac formula

Exercise 12.1 *(Running payoffs)* [⋆] Work in the setting of Theorem 12.1, and let $f : [0,T] \times I \to \mathbb{R}$ be continuous. Suppose that $v \in C^{1,2}([0,T) \times I)$ solves

$$\partial_t v + \mathcal{L}_t v - kv + f = 0 \qquad \text{on } [0,T) \times I,$$

with the terminal condition of the theorem (Equation (12.3)), and that the solution X on $[t,T]$ satisfies $\mathbb{P}(X_T \in E) = 0$, the integrability condition (Equation (12.4)), and $\mathbb{E}\int_t^T |f(u,X_u)|\,\mathrm{d}u < \infty$.

(a) Show that, almost surely,

$$v(t,X_t) = \mathbb{E}\Big[\int_t^T e^{-\int_t^u k(s,X_s)\,\mathrm{d}s} f(u,X_u)\,\mathrm{d}u + e^{-\int_t^T k(s,X_s)\,\mathrm{d}s} h(X_T)\,\Big|\,\mathcal{F}_t\Big].$$

Hint: which process has no drift now?

(b) In the Black–Scholes model, a contract pays its holder money continuously, at the rate δS_u per unit of time over $[0,T]$, with $\delta > 0$, and nothing at T. Show that $v(t,x) = \delta x(T-t)$ satisfies the hypotheses of (a) with $k \equiv r$, $f(t,x) = \delta x$ and $h = 0$, and deduce the price of the contract at time t. Check the answer by computing the expectation directly.

(c) Find the price at time t of a contract that pays at the constant rate $c > 0$ over $[t,T]$, by solving the equation of (a).

Exercise 12.2 *(Parity from uniqueness)* [⋆] Work in the local volatility model of Theorem 12.4, with a volatility function σ satisfying the conditions (Equation (12.6)).

(a) Show that $w(t,x) = x - Ke^{-r(T-t)}$ solves the pricing equation (Equation (12.7)) with $w(T,x) = x - K$, whatever σ is.

(b) Let c and p be solutions of the pricing equation in $C^{1,2}([0,T) \times (0,\infty))$, continuous on $[0,T] \times (0,\infty)$, with $|c(t,x)| + |p(t,x)| \le C(1+x^2)$ and terminal values $(x-K)^+$ and $(K-x)^+$. Without computing c or p, show that $c - p = w$. Deduce that the deltas of the call and the put differ by 1 and that their gammas are equal.

(c) Let v be a solution as in (b), with terminal value a payoff g, and let $a, b \in \mathbb{R}$. Show that $v + ae^{-r(T-t)} + bx$ is the only solution with quadratic growth and terminal value $g(x) + a + bx$, in every such model. Which static portfolio replicates $a + bS_T$?

Exercise 12.3 *(A quadratic potential)* [⋆⋆] Let $\lambda > 0$, $c = \sqrt{2\lambda}$, and for $(t, x) \in [0, T] \times \mathbb{R}$ let

$$u(t,x) = \mathbb{E}\Big[\exp\Big(-\lambda \int_t^T (x + W_s - W_t)^2 \, \mathrm{d}s\Big)\Big].$$

(a) Look for a solution of $\partial_t v + \frac{1}{2}\partial_{xx} v - \lambda x^2 v = 0$ with $v(T, x) = 1$ of the form $v(t, x) = \exp\big(-\frac{1}{2}A(t)x^2 - B(t)\big)$. Show that $A' = A^2 - 2\lambda$ and $B' = -\frac{1}{2}A$ with $A(T) = B(T) = 0$, and that $A(t) = c\tanh(c(T-t))$, $B(t) = \frac{1}{2}\log\cosh(c(T-t))$.

(b) Use Theorem 12.1 to show that $u = v$. In particular,

$$\mathbb{E}\Big[\exp\Big(-\lambda \int_0^T W_s^2 \, \mathrm{d}s\Big)\Big] = \cosh\big(\sqrt{2\lambda}\, T\big)^{-1/2}.$$

(c) Let $R = \int_0^T W_s^2 \, \mathrm{d}s$ and $\phi(\lambda) = \mathbb{E}[e^{-\lambda R}]$ for $\lambda \geq 0$. You may use that $\mathbb{E}[R] = -\phi'(0+)$ and $\mathbb{E}[R^2] = \phi''(0+)$, the one-sided derivatives at 0 of the formula of (b). Compute $\mathbb{E}[R]$ and $\mathrm{Var}(R)$, and check both directly, using $\mathrm{Cov}(W_s^2, W_u^2) = 2\min(s, u)^2$ (Exercise 5.1).

(d) Now let $\mu > 0$: the discount rate $k(x) = -\mu x^2$ is not bounded below. Show that $\mathbb{E}\big[\exp(\mu \int_0^T W_s^2 \, \mathrm{d}s)\big] = \infty$ for $T \geq \sqrt{3/(2\mu)}$. *Hint:* use the Cauchy–Schwarz inequality $\big(\int_0^T W_s \, \mathrm{d}s\big)^2 \leq T \int_0^T W_s^2 \, \mathrm{d}s$, and the fact that $\int_0^T W_s \, \mathrm{d}s$ is normal with mean 0 and variance $T^3/3$. What happens to the function v of (a) if c is replaced by $i\sqrt{2\mu}$, and why can Theorem 12.1 not be applied to it?

(e) Estimate $\mathbb{E}\big[\exp(-\int_0^1 W_s^2 \, \mathrm{d}s)\big]$ by Monte Carlo, discretising the integral on a grid, with a 95% confidence interval, and compare with (b).

Exercise 12.4 *(Pay-later options: value and hedge)* [⋆⋆] In the Black–Scholes model, let P be the fee of the pay-later call of Theorem 12.12, c the call function of Theorem 6.11, and $D(t, x) = e^{-r\tau}N(d_-)$ the digital function. The value of the holder's position at time $t < T$ is $V(t, S_t)$ with $V = c - PD$. Take $S_0 = K = 100$, $r = 5\%$, $\sigma = 20\%$ and $T = 1$ for the numbers.

(a) On which event does the holder end up paying more than she receives? Compute its probability under $\mathbb{Q}$.

(b) Fix $t < T$. Show that $V(t, x) < 0$ for all small enough $x > 0$ and $V(t, x) > 0$ for all large enough x. *Hint:* write $V/D = c/D - P$, and use the identity $Ke^{-r\tau}N'(d_-) = xN'(d_+)$ of Theorem 6.14 and the fact that $|y|N(y)/N'(y) \to 1$ as $y \to -\infty$. Interpret the sign of V after a fall of the stock.

(c) Write the delta of V and show that, at $x = K$, it tends to $-\infty$ as $\tau \downarrow 0$. Compute it one week ($\tau = 1/52$) and one day ($\tau = 1/365$) before maturity, and explain what a seller who delta-hedges the contract must do near maturity when the stock is close to the strike.

Exercise 12.5 *(The hedge of a digital option)* [⋆⋆⋆] In the Black–Scholes model under $\mathbb{Q}$, let $D(t, x) = e^{-r\tau}N(d_-)$ be the price function of the digital call with strike K, $\Delta_t = \partial_x D(t, S_t)$ for $t < T$, and $p = \mathbb{Q}(S_T > K) = N(d_-(T, S_0))$.

(a) Show that $e^{-ru}\sigma S_u\Delta_u = e^{-rT}N'(d_-(T-u,S_u))/\sqrt{T-u}$, and that $M_t = e^{-rt}D(t,S_t)$ satisfies $M_t = D(0,S_0) + \int_0^t e^{-ru}\sigma S_u\Delta_u\,\mathrm{d}\widetilde{W}_u$ for $t<T$ and $M_t \to e^{-rT}\mathbf{1}_{\{S_T>K\}}$ as $t\uparrow T$, almost surely.

(b) Show that

$$\mathbb{E}^{\mathbb{Q}}\int_0^T \left(e^{-ru}\sigma S_u\Delta_u\right)^2 \mathrm{d}u = e^{-2rT}p(1-p).$$

Hint: the Itô isometry on $[0,T']$, then $T'\uparrow T$. Deduce that $\int_0^T \Delta_u^2\,\mathrm{d}u<\infty$ almost surely, although $\partial_x D(t,K)\to\infty$ as $t\uparrow T$ (Theorem 12.11).

(c) Show that (b) is the identity

$$N(d)(1-N(d)) = \frac{1}{2\pi}\int_0^{\pi/2}\exp\Big(-\frac{d^2}{1+\sin\theta}\Big)\,\mathrm{d}\theta, \qquad d = d_-(T,S_0),$$

by computing the expectation of the integrand at each time u, and check it for $d=0$. *Hint:* if $Y\sim\mathcal{N}(m,v)$, then $\mathbb{E}[e^{-Y^2}] = (1+2v)^{-1/2}e^{-m^2/(1+2v)}$; then substitute $u = T\sin\theta$.

(d) The gamma does not share this property. Using $\partial_{xx}D(t,x) = -e^{-r\tau}N'(d_-)\,d_+/(x^2\sigma^2\tau)$, show that $\mathbb{E}^{\mathbb{Q}}\int_0^T S_u^4(\partial_{xx}D(u,S_u))^2\,\mathrm{d}u = \infty$. *Hint:* for u close to T, bound the integrand from below on the event $\{\frac12 \le d_-(T-u,S_u)\le 1\}$, and show that this event has probability at least a constant times $\sqrt{T-u}$. Compare with the expression for the variance of the error of a discretely rebalanced hedge in Theorem 10.11.

Exercise 12.6 *(All strikes from one forward equation)* [⋆⋆⋆] This is a computational exercise. Take the local volatility model of Figure 12.1: $\sigma(t,x) = 0.25 - 0.12\tanh(2\log(x/100))$, $r = 3\%$, $S_0 = 100$, $T = 1$.

(a) Write Kolmogorov's forward equation (Theorem 12.8) for the density $q(s,\eta)$ of $\log S_s$ under $\mathbb{Q}$.

(b) Solve it numerically for $\eta\in[\log 5, \log 2000]$ and $s\in[s_0,T]$, starting at $s_0 = 0.005$ from the normal density of $\log S_{s_0}$ in the Black–Scholes model with volatility $\sigma(0,S_0)$, with any stable scheme (for instance Crank–Nicolson in s and central differences in η, with $q=0$ at both ends). Check numerically that $\int q(T,\eta)\,\mathrm{d}\eta = 1$ and $\int e^{\eta}q(T,\eta)\,\mathrm{d}\eta = S_0e^{rT}$, and explain why both identities should hold.

(c) From $q(T,\cdot)$, compute the call prices $e^{-rT}\int(e^{\eta}-K)^+q(T,\eta)\,\mathrm{d}\eta$ for $K = 70, 80, \dots, 140$. Compare them with solutions of the pricing equation (Equation (12.7)), one for each strike, and with the Black–Scholes prices at $\sigma = 25\%$. For the pricing equation you may use the function `solve_pricing_pde` of the course code (`code/src/ieor222/feynmankac.py`). How many equations does each route solve for 8 strikes? For 8 strikes and 12 maturities?

(d) Plot the density of S_T against the lognormal density of the Black–Scholes model with $\sigma = 25\%$, and relate the differences to the comparison of prices in (c).

Chapter 13

Exotic options

Unless an exercise says otherwise, we are in the setting of Section 13.1: under $\mathbb{Q}$, $S_t = S_0 e^{\sigma X_t}$ with $X_t = \widetilde{W}_t + \nu t$, $\nu = r/\sigma - \sigma/2$, prices are given by Equation (13.2), and $c(x;k)$, $\delta(x;k)$ are the prices at time 0 of the call and of the digital call with strike k and maturity T when the stock is at x, as in Theorem 13.5. N and N' are the standard normal distribution function and density.

Exercise 13.1 *(One-touch options)* [⋆] Let $B > S_0$. A *one-touch* option pays 1 at T if $\overline{S}_T \geq B$, and nothing otherwise; a *no-touch* option pays 1 if $\overline{S}_T < B$.

(a) Show that the one-touch option costs

$$e^{-rT} N\big(d_-(T, S_0)\big) + \Big(\frac{B}{S_0}\Big)^{2r/\sigma^2 - 1} e^{-rT} N\Big(- d_-\Big(T, \frac{B^2}{S_0}\Big)\Big),$$

where d_- is computed with the strike B. Interpret the two terms as a digital call and the image of a digital put, in the sense of the discussion after Theorem 13.5. What does the no-touch option cost?

(b) Show that when $r = \sigma^2/2$ the one-touch option costs exactly twice the digital call paying 1 if $S_T \geq B$. Explain this with the reflection principle.

(c) Take $S_0 = 100$, $B = 110$, $r = 2\%$, $\sigma = 25\%$ and $T = \frac{1}{2}$. Compute the prices of the one-touch option and of the digital call. Is the rule of thumb "a one-touch is worth twice the digital" accurate here?

Exercise 13.2 *(Hitting times)* [⋆⋆] Let W be a Brownian motion with a filtration, $\nu \in \mathbb{R}$, $X_t = W_t + \nu t$, $m > 0$, and $\tau_m = \inf\{t \geq 0 : X_t = m\}$ ($\inf \emptyset = \infty$).

(a) Let $\nu = 0$. Show that $\mathbb{P}(\tau_m \leq t) = 2N(-m/\sqrt{t})$ for $t > 0$, that τ_m has the density $\frac{m}{\sqrt{2\pi}\, t^{3/2}} e^{-m^2/(2t)}$, and that τ_m has the law of m^2/Z^2 for $Z \sim \mathcal{N}(0,1)$.

(b) Show that $\mathbb{P}(\tau_m < \infty) = 1$ if $\nu \geq 0$ and $\mathbb{P}(\tau_m < \infty) = e^{2\nu m}$ if $\nu < 0$.

(c) Let $\lambda > 0$ and $\alpha = \sqrt{\nu^2 + 2\lambda} - \nu$. Show that $e^{\alpha X_t - \lambda t}$ is a martingale, and deduce that $\mathbb{E}\big[e^{-\lambda \tau_m} \mathbf{1}_{\{\tau_m < \infty\}}\big] = e^{-\alpha m}$. You may use that a stopped martingale of the form

$\exp(\alpha W_t - \frac{\alpha^2}{2}t)$ is a martingale, as in the proof of Theorem 13.12.

(d) A *perpetual one-touch* option pays 1 at the first time τ the stock price reaches $B > S_0$, if it ever does; as in Section 13.6, its price is $\mathbb{E}^{\mathbb{Q}}\big[e^{-r\tau}\mathbf{1}_{\{\tau<\infty\}}\big]$. Show that this is S_0/B, whatever $r > 0$ and σ. Find a portfolio of shares, bought at time 0 and sold at the hitting time, that replicates it.

Exercise 13.3 *(The delta of an up-and-out call)* [⋆⋆] Let $K < B$ and $S_0 < B$, and let u be the function of Theorem 13.5(i).

(a) Consider any arbitrage-free market, not necessarily the Black–Scholes one, in which calls with strikes K and B and a digital call with strike B, all with maturity T, trade at the prices $C(K)$, $C(B)$ and $D(B)$; the digital pays 1 if $S_T \geq B$. Show that an up-and-out call with strike K and barrier B, if it trades, costs at most $C(K) - C(B) - (B-K)D(B)$. You may use Theorem 3.3.

(b) In the Black–Scholes model, let $UO(x)$ be the price of Theorem 13.5(i) as a function of $x = S_0 \in (0, B)$. Show that $UO(x) \to 0$ as $x \uparrow B$ and that the left derivative at the barrier is $UO'(B-) = 2u'(B) + \theta\, u(B)/B$, with $\theta = 2r/\sigma^2 - 1$.

(c) Show that, with K, B, r, σ fixed, $\sqrt{T}\,UO'(B-) \to -(B-K)\sqrt{2/\pi}/(\sigma B)$ as $T \downarrow 0$.

(d) Why must the delta be negative close to the barrier? A trader is short this option, with S_t just below B, a few days before maturity. Describe her hedge, and what happens to it if the stock price crosses the barrier.

Exercise 13.4 *(Fixed-strike lookback calls)* [⋆⋆] A *fixed-strike lookback call* pays $(\overline{S}_T - K)^+$, with $K > 0$. Let $r \neq 0$, and let LP_0 be the price of Theorem 13.8.

(a) Let $K \leq S_0$. Show that the fixed-strike lookback call costs $LP_0 + S_0 - Ke^{-rT}$.

(b) Let $K > S_0$ and $k = \frac{1}{\sigma}\log(K/S_0)$. Show that the price is $e^{-rT}S_0 \int_k^\infty \sigma e^{\sigma m}\, \mathbb{Q}(\overline{X}_T \geq m)\, \mathrm{d}m$, and that it equals

$$c(S_0; K) + S_0\,\frac{\sigma^2}{2r}\Big[N(d_+) - e^{-rT}\Big(\frac{K}{S_0}\Big)^{2r/\sigma^2} N(e_-)\Big],$$

where $d_+ = d_+(T, S_0)$ is computed with the strike K and $e_- = \big(\log(S_0/K) - (r - \frac{\sigma^2}{2})T\big)/(\sigma\sqrt{T})$.

(c) Check that the formulas of (a) and (b) agree at $K = S_0$, and that the price exceeds the price of the call with the same strike, as it must by Exercise 3.1(d).

Exercise 13.5 *(Asian options: parity and bounds)* [⋆] Let $A_T = Y_T/T = \frac{1}{T}\int_0^T S_u\,\mathrm{d}u$, $K > 0$, and γ as in Theorem 13.11.

(a) Show that $e^{-rT}\mathbb{E}^{\mathbb{Q}}[A_T] = \gamma(0)S_0$, and deduce a put–call parity for Asian options: the Asian call minus the Asian put paying $(K - A_T)^+$ costs $\gamma(0)S_0 - Ke^{-rT}$.

(b) Show that, in any model in which $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale, the Asian call costs at least $\big(\gamma(0)S_0 - Ke^{-rT}\big)^+$, with equality when $K \leq 0$ (allow $K \leq 0$ here).

(c) Show that $(A_T - K)^+ \le \frac{1}{T}\int_0^T (S_t - K)^+ \,\mathrm{d}t$, and deduce that the arithmetic Asian call costs at most $\frac{1}{T}\int_0^T e^{-r(T-t)} C(t)\,\mathrm{d}t$, where $C(t)$ is the price at time 0 of the call with strike K and maturity t. Conclude that, if $r \ge 0$, the Asian call is cheaper than the European call with maturity T.

Exercise 13.6 *(From Večeř's equation to the Asian equation)* [⋆⋆] Let $\tilde{g}$ be as in Theorem 13.11(iii), let $\pi(t,x,y) = e^{-r(T-t)}(y/T - K) + \gamma(t)x$, and $w(t,x,y) = x\,\tilde{g}(t, \pi(t,x,y)/x)$ for $x > 0$, $y \in \mathbb{R}$, $t \in [0,T]$.

(a) Show that $w(T,x,y) = (y/T - K)^+$.

(b) Show that w solves the Asian equation Equation (13.7) on $[0,T) \times (0,\infty) \times \mathbb{R}$.

(c) Show that $\partial_x w(t,x,y) = \tilde{g}(t,z) + (\gamma(t) - z)\,\partial_z \tilde{g}(t,z)$ with $z = \pi(t,x,y)/x$, and compare with the two hedges of Theorem 13.10 and Theorem 13.11.

(d) Let $y \ge KT$. Without solving any equation, show that the price of the Asian call is $e^{-r(T-t)}(y/T - K) + \gamma(t)x$ at time t when $S_t = x$ and $Y_t = y$, and give the hedge.

Exercise 13.7 *(Perpetual American calls on a dividend-paying stock)* [⋆⋆⋆] The stock pays a continuous dividend yield $q > 0$, and $r > 0$. By Theorem 11.14, under $\mathbb{Q}$, $S_t = S_0 \exp\big(\sigma\widetilde{W}_t + (r - q - \frac{\sigma^2}{2})t\big)$. The perpetual American call is worth $v(x) = \sup_\tau \mathbb{E}^{\mathbb{Q}}\big[e^{-r\tau}(S_\tau - K)^+ \mathbf{1}_{\{\tau<\infty\}}\big]$, the supremum over stopping times with values in $[0,\infty]$, for $S_0 = x$.

(a) Adapting Exercise 6.2(b) to the dividend yield, show that $e^{-rt}S_t^p$ is a $\mathbb{Q}$-martingale if and only if $\frac{\sigma^2}{2}p(p-1) + (r-q)p - r = 0$, and that this equation has one root $p_+ > 1$ and one root $p_- < 0$.

(b) For $L \ge x$, let $\tau^L = \inf\{t \ge 0 : S_t \ge L\}$. Show that $\mathbb{E}^{\mathbb{Q}}\big[e^{-r\tau^L}\mathbf{1}_{\{\tau^L<\infty\}}\big] = (x/L)^{p_+}$.

(c) Let $L^* = \frac{p_+}{p_+ - 1}K$. Show that $v(x) = (L^* - K)(x/L^*)^{p_+}$ for $x \le L^*$ and $v(x) = x - K$ for $x \ge L^*$.

(d) Show that, as $q \downarrow 0$ with the other parameters fixed, $L^* \to \infty$ and $v(x) \to x$. Interpret with Theorem 4.13(iv).

Exercise 13.8 *(Discretely monitored lookbacks)* [⋆⋆] This is a computational exercise. Take $S_0 = 100$, $r = 5\%$, $\sigma = 30\%$ and $T = 1$, and let V_m be the price of the lookback put paying $\max_{0\le i\le m} S_{iT/m} - S_T$, in which the maximum is taken over $m+1$ equally spaced dates only.

(a) Estimate V_m by Monte Carlo for $m = 12$, 52 and 252, with 95% confidence intervals, and compare with the price V of Theorem 13.8.

(b) How does $V - V_m$ appear to depend on m?

(c) M. Broadie, P. Glasserman and S. Kou (*Finance and Stochastics*, 1999) showed that $V_m + S_0 = (V + S_0)(1 - \beta\sigma\sqrt{T/m}) + o(1/\sqrt{m})$, with $\beta \approx 0.5826$ as in Theorem 13.7. Use this to correct your estimates, and compare with V.

(d) Why is the discretely monitored lookback put cheaper than the continuously monitored one, while the discretely monitored up-and-out call of Figure 13.2 is more expensive?

Chapter 14

Short-rate models

Exercise 14.1 *(Yields, forward rates and bootstrapping)* [⋆]

(a) Let $T \mapsto P(t,T)$ be continuously differentiable and positive on $[t,T^*]$. Show that $f(t,T) = Y(t,T) + (T-t)\,\partial_T Y(t,T)$ for $t < T$. Deduce that the forward curve lies above the yield curve where the yield curve is increasing, and below it where it is decreasing.

(b) Three bonds with face value 100 pay annual coupons and mature in 1, 2 and 3 years: the first pays 2% of the face value at the end of each year, the second 3%, the third 4%, and each repays the face value at maturity. Their prices are 99.20, 99.60 and 100.30. Assuming that a bond paying $c_1, \dots, c_n$ at times $T_1 < \cdots < T_n$ is worth $\sum_i c_i P(0,T_i)$, find $P(0,1)$, $P(0,2)$ and $P(0,3)$, the yields $Y(0,k)$ and the forward rates $F(0;1,2)$ and $F(0;2,3)$ of Equation (14.1).

(c) Check that $kY(0,k) = \sum_{j=1}^{k} F(0;j-1,j)$ (with $F(0;0,1) = Y(0,1)$), and explain this identity with the strategy described after Equation (14.1).

(d) What is the price of a 3-year bond with face value 100 and an annual coupon of 5%? Why must it be this price?

Exercise 14.2 *(The term premium in the Vasicek model)* [⋆] Let r follow the Vasicek model of Theorem 14.8 under $\mathbb{Q}$, with parameters $\kappa > 0$, θ and $\sigma > 0$, and let $\lambda \in \mathbb{R}$ be a constant market price of risk: as in Theorem 14.4, $\mathbb{P}$ is the measure under which $W_t = \widetilde{W}_t - \lambda t$ is a Brownian motion.

(a) Show that under $\mathbb{P}$ the short rate is an Ornstein–Uhlenbeck process with the same κ and σ and the long-run level $\theta_{\mathbb{P}} = \theta + \sigma\lambda/\kappa$.

(b) Show that under $\mathbb{P}$ the bond maturing at T has expected instantaneous return $r_t - \lambda\sigma B(T-t)$ at time $t < T$. For which sign of λ do long bonds earn more than short ones on average? What is the ratio of the expected excess return to the volatility?

(c) The *term premium* is $\mathrm{TP}(T) = Y(0,T) - \frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{P}}[r_t]\,\mathrm{d}t$: the yield in excess of the average

short rate expected under $\mathbb{P}$. Show that

$$\mathrm{TP}(T) = (R_\infty - \theta_{\mathbb{P}})\Big(1 - \frac{B(T)}{T}\Big) + \frac{\sigma^2 B(T)^2}{4\kappa T},$$

which does not depend on r_0, and find its limit as $T \to \infty$. Identify the two effects that it combines.

(d) Take $\kappa = 0.3$, $\theta_{\mathbb{P}} = 4\%$, $\sigma = 1\%$, $\lambda = -0.3$ and $r_0 = 3\%$. Compute θ, R_∞, $Y(0,10)$, $\frac{1}{10}\int_0^{10} \mathbb{E}^{\mathbb{P}}[r_t]\,\mathrm{d}t$, $\mathrm{TP}(10)$, and the expected excess return of the 10-year bond at time 0.

Exercise 14.3 *(Vasicek bond prices from the integrated rate)* [⋆⋆] Let r follow the Vasicek model of Theorem 14.8, $0 \le t < T$ and $\tau = T - t$. You may use that, for deterministic $g, h \in L^2[t,T]$, $\mathbb{E}\big[\int_t^T g\,\mathrm{d}\widetilde{W}\int_t^T h\,\mathrm{d}\widetilde{W}\big] = \int_t^T g(u)h(u)\,\mathrm{d}u$.

(a) Integrate the equation of r from t to T and use Theorem 9.8(i) to show that

$$\int_t^T r_u\,\mathrm{d}u = \theta\tau + (r_t - \theta)B(\tau) + \sigma\int_t^T B(T-u)\,\mathrm{d}\widetilde{W}_u.$$

(b) Deduce that, given $\mathcal{F}_t$, $\int_t^T r_u\,\mathrm{d}u$ is normal with mean $m = \theta\tau + (r_t-\theta)B(\tau)$ and variance $v = \sigma^2\int_0^\tau B(u)^2\,\mathrm{d}u$, and recover Theorem 14.8(i) from $\mathbb{E}^{\mathbb{Q}}[e^{-X}] = e^{-\mathbb{E}X + \mathrm{Var}\,X/2}$ for a normal X.

(c) Show that $Y(t,T) < m/\tau$, the average short rate expected under $\mathbb{Q}$, and that the difference $v/(2\tau)$ increases to $\sigma^2/(2\kappa^2)$ as $\tau \to \infty$. Interpret R_∞. For $\kappa = 0.15$, $\theta = 5\%$, $\sigma = 2.5\%$, compute the difference in basis points for $\tau = 1$, 10 and 30 years.

(d) Show that, given $\mathcal{F}_t$, the pair $\big(r_{t+h}, \int_t^{t+h} r_u\,\mathrm{d}u\big)$ is Gaussian with covariance $\sigma^2 B(h)^2/2$, and give its means and variances. Explain how to simulate $\big(r_{t_k}, \int_0^{t_k} r_u\,\mathrm{d}u\big)_{k\le n}$ exactly on a grid.

Exercise 14.4 *(Moments of the CIR rate)* [⋆⋆] Let r be the CIR rate of Theorem 14.9 with $\kappa, \theta, \sigma > 0$ and $r_0 \ge 0$. You may use that $\mathbb{E}^{\mathbb{Q}}\big[\sup_{s\le t} r_s\big] < \infty$ for every t.

(a) Compute $\mathrm{d}(e^{\kappa t} r_t)$ and deduce that $r_t = \theta + (r_0 - \theta)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}\sqrt{r_s}\,\mathrm{d}\widetilde{W}_s$.

(b) Show that the stochastic integral in (a) has mean zero, and find $\mathbb{E}^{\mathbb{Q}}[r_t]$. Compare with the Vasicek model.

(c) Show that $\mathrm{Var}(r_t) = \frac{\sigma^2 r_0}{\kappa}\big(e^{-\kappa t} - e^{-2\kappa t}\big) + \frac{\sigma^2\theta}{2\kappa}\big(1 - e^{-\kappa t}\big)^2$.

(d) Find the limits of $\mathbb{E}^{\mathbb{Q}}[r_t]$ and $\mathrm{Var}(r_t)$ as $t \to \infty$, and check them against the stationary gamma law of Exercise 9.5(d) when $2\kappa\theta \ge \sigma^2$.

(e) A Vasicek model with the same κ and θ has the same stationary mean and variance if its volatility is $\sigma_V = \sigma\sqrt{\theta}$. For $\kappa = 0.5$, $\theta = 4\%$, $\sigma = 10\%$, compute the stationary probability of a negative rate in this Vasicek model, and the Feller ratio of the CIR model. What do the two models agree on, and what not?

Exercise 14.5 *(The Hull–White model)* [⋆⋆] Let $\kappa > 0$, $\sigma > 0$, $r_0 \in \mathbb{R}$, $\beta : [0, T^*] \to \mathbb{R}$ continuous, and let r solve $\mathrm{d}r_t = (\beta(t) - \kappa r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}\widetilde{W}_t$.

(a) Use Theorem 14.5 to show that $P(t,T) = \exp\big(A(t,T) - B(T-t)r_t\big)$ with $B(\tau) = (1-e^{-\kappa\tau})/\kappa$ and $A(t,T) = -\int_t^T \beta(s)B(T-s)\,\mathrm{d}s + \frac{\sigma^2}{2}\int_0^{T-t} B(u)^2\,\mathrm{d}u$.

(b) Show that $f(0,T) = e^{-\kappa T}r_0 + \int_0^T \beta(s)e^{-\kappa(T-s)}\,\mathrm{d}s - \frac{\sigma^2}{2}B(T)^2$.

(c) Let f^M be a continuously differentiable forward curve and $P^M(0,T) = \exp(-\int_0^T f^M)$. Show that $P(0,T) = P^M(0,T)$ for all $T \le T^*$ if and only if $r_0 = f^M(0)$ and

$$\beta(t) = (f^M)'(t) + \kappa f^M(t) + \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}), \qquad t \le T^*.$$

Hint: $g(T) = f(0,T) + \frac{\sigma^2}{2}B(T)^2$ solves a linear differential equation.

(d) Check that the formula of (c) tends to that of Theorem 14.7(ii) as $\kappa \to 0$.

(e) Suppose the market curve is flat, $f^M \equiv \varphi$. Find β. Can a Vasicek model (constant $\beta = \kappa\theta$) with $\sigma > 0$ reproduce a flat curve?

Exercise 14.6 *(Caplets are bond puts)* [⋆⋆] Let r be a short-rate model, $0 < \delta$ and $T + \delta \le T^*$. The simple rate for $[T, T+\delta]$ fixed at T is $L = (1/P(T,T+\delta) - 1)/\delta$: one dollar invested at T in bonds maturing at $T+\delta$ grows to $1 + \delta L$.

(a) Show that a claim paying an $\mathcal{F}_T$-measurable amount $X \ge 0$ at time $T+\delta$ is worth $X\,P(T,T+\delta)$ at time T, in the sense of Theorem 11.8. Deduce that the claim paying δL at $T+\delta$ is worth $P(0,T) - P(0,T+\delta)$ at time 0, and find the fixed rate k that makes the swap of δL against δk, both paid at $T+\delta$, worth zero.

(b) A *caplet* with strike $k > 0$ pays $\delta(L-k)^+$ at $T+\delta$. Show that it is worth $(1+\delta k)\big(\frac{1}{1+\delta k} - P(T,T+\delta)\big)^+$ at time T: it is $1+\delta k$ puts on the bond maturing at $T+\delta$, with expiry T.

(c) In the Vasicek model, give the price of the caplet at time 0. Compute it, per unit of notional and in basis points, for $\kappa = 0.15$, $\theta = 5\%$, $\sigma = 2.5\%$, $r_0 = 3\%$, $T = 1$, $\delta = 0.5$ and $k = 4\%$, together with the forward simple rate of (a).

Exercise 14.7 *(Options on coupon bonds)* [⋆⋆⋆] This exercise is partly computational. In the Vasicek model, a coupon bond pays $c_i > 0$ at times $T < T_1 < \cdots < T_n \le T^*$, and a European call with strike $K > 0$ and expiry T pays $\big(\sum_i c_i P(T,T_i) - K\big)^+$ at T. Write $P(T,T_i) = \Pi_i(r_T)$, with $\Pi_i(x) = \exp\big(A(T_i - T) - B(T_i - T)x\big)$.

(a) Show that $x \mapsto \sum_i c_i \Pi_i(x)$ is continuous and strictly decreasing on $\mathbb{R}$, with limits $+\infty$ and 0, so that there is a unique r^* with $\sum_i c_i \Pi_i(r^*) = K$.

(b) Let $K_i = \Pi_i(r^*)$. Show that $\big(\sum_i c_i P(T,T_i) - K\big)^+ = \sum_i c_i \big(P(T,T_i) - K_i\big)^+$ (*Jamshidian's decomposition*), and deduce the price of the call at time 0 from Theorem 14.10(iii).

(c) Take $\kappa = 0.15$, $\theta = 5\%$, $\sigma = 2.5\%$, $r_0 = 3\%$, $T = 1$, a bond paying 5 at times 2 and 3 and 105 at time 4, and $K = 100$. Compute r^*, the K_i and the price of the call. Check the price by Monte Carlo with 2×10^5 exact draws of $\big(r_1, \int_0^1 r_u\,\mathrm{d}u\big)$ (Exercise 14.3(d)), with a 95% confidence interval, and compute the put by parity.

(d) Where does the argument use that the model has a single factor? Would it apply if $r_t = X_t + Y_t$ for two independent Ornstein–Uhlenbeck processes X and Y?

Exercise 14.8 *(When the Feller condition fails)* [⋆⋆⋆] Let r be the CIR rate of Theorem 14.9, with $\kappa, \theta, \sigma > 0$ and $r_0 > 0$, let $\nu = 2\kappa\theta/\sigma^2$, and for $c \geq 0$ let $\tau_c = \inf\{t \geq 0 : r_t = c\}$ ($\inf \emptyset = \infty$). Define, for $x > 0$,

$$s(x) = \int_1^x y^{-\nu} e^{2\kappa y/\sigma^2}\, \mathrm{d}y.$$

(a) Show that $\frac{1}{2}\sigma^2 x\, s''(x) + \kappa(\theta - x)s'(x) = 0$ for $x > 0$.

(b) Let $0 < a < r_0 < b$ and $\tau = \tau_a \wedge \tau_b$. You may use that, for $0 \leq a < r_0 < b$, $\tau_a \wedge \tau_b < \infty$ almost surely: the rate leaves every bounded interval. Show that $\big(s(r_{t\wedge\tau})\big)_{t\geq 0}$ is a bounded martingale, and deduce that

$$\mathbb{Q}(\tau_a < \tau_b) = \frac{s(b) - s(r_0)}{s(b) - s(a)}.$$

(c) Let $\nu < 1$. Show that $s(0+) = \lim_{x\downarrow 0} s(x)$ is finite and that $\mathbb{Q}(\tau_0 < \tau_b) = \frac{s(b)-s(r_0)}{s(b)-s(0+)} > 0$. *Hint:* the events $\{\tau_a < \tau_b\}$ decrease as $a \downarrow 0$; identify their intersection.

(d) Still with $\nu < 1$, show that $\mathbb{Q}(\tau_0 < \infty) = 1$.

(e) Let $\nu \geq 1$. Show that $s(0+) = -\infty$, and deduce Theorem 14.9(ii) again.

Chapter 15

Implied volatility and the smile

Exercise 15.1 *(Variance and volatility swaps)* [⋆] A *volatility swap* pays $\sqrt{RV} - K_{\text{vol}}$ at T, where $RV = \frac{1}{T}\int_0^T \sigma_t^2 \, \mathrm{d}t$ is the realized variance of Theorem 15.13, and $K_{\text{vol}} = \mathbb{E}^{\mathbb{Q}}[\sqrt{RV}]$ makes it worth 0. Let $K_{\text{var}} = \mathbb{E}^{\mathbb{Q}}[RV]$.

(a) Show that $K_{\text{vol}} \le \sqrt{K_{\text{var}}}$, with equality if and only if RV is almost surely constant.

(b) Write $RV = K_{\text{var}}(1+\varepsilon)$ and expand $\sqrt{1+\varepsilon}$ to second order to show that, when $\operatorname{Var}(RV)$ is small, $K_{\text{vol}} \approx \sqrt{K_{\text{var}}} - \operatorname{Var}(RV)/(8K_{\text{var}}^{3/2})$.

(c) Evaluate the approximation for $K_{\text{var}} = 0.04$ and a standard deviation of RV of 0.01, and compare with the exact value when RV takes the values 0.03 and 0.05 with probability $\frac{1}{2}$ each. Why can the volatility swap not, in general, be replicated by a static strip of options and a dynamic position in the stock, as the variance swap is?

Exercise 15.2 *(Normal and lognormal implied volatilities)* [⋆⋆] Take $r = q = 0$, so that $F_T = S_0$, and fix $T > 0$. In the Bachelier model of Theorem 15.5, the call with strike K costs $C_{\text{N}}(K) = (S_0 - K)N(d) + \sigma_{\text{N}}\sqrt{T}\,N'(d)$, $d = (S_0 - K)/(\sigma_{\text{N}}\sqrt{T})$.

(a) At $K = S_0$, show that $C_{\text{N}}(S_0) = \sigma_{\text{N}}\sqrt{T/(2\pi)}$. If the same price is quoted with a normal implied volatility σ_{N} and a Black–Scholes implied volatility σ_{BS}, show that $0 \le \sigma_{\text{BS}} - \sigma_{\text{N}}/S_0 \le \sigma_{\text{BS}}^3 T/24$. You may use Theorem 6.16. How good is σ_{N}/S_0 as an estimate of σ_{BS} for $\sigma_{\text{BS}} = 20\%$ and $T = 1$?

(b) Now fix $\sigma_{\text{N}} > 0$ and let $\Sigma(K)$ be the Black–Scholes implied volatility of $C_{\text{N}}(K)$. Show that Σ is differentiable and, with $s = \Sigma(S_0)\sqrt{T}$,

$$\Sigma'(S_0) = -\frac{N(s/2) - \frac{1}{2}}{S_0\sqrt{T}\,N'(s/2)} < 0, \qquad \Sigma'(S_0) \approx -\frac{\Sigma(S_0)}{2S_0} \quad \text{for small } s.$$

(c) Compute $\Sigma(K)$ for $S_0 = 100$, $\sigma_{\text{N}} = 20$, $T = 1$ and $K = 70, 80, \dots, 130$, and compare with $\sigma_{\text{N}}/\sqrt{S_0 K}$. What does a trader who believes in the Bachelier model see in the Black–Scholes smile?

Exercise 15.3 *(The wings of the smile)* [⋆⋆] Let the call prices at one maturity come from a pricing measure $\mathbb{Q}$, and let $w(k)$ be their total implied variance, so that $d_+(k) = -k/\sqrt{w(k)} + \sqrt{w(k)}/2 \to -\infty$ as $k \to \infty$ (Theorem 15.6).

(a) Show that $w(k) < 2k$ for all k large enough, so that $\limsup_{k\to\infty} w(k)/k \le 2$.

(b) Suppose $\mathbb{Q}(S_T = 0) = 0$. Show that $P(K,T)/(Ke^{-rT}) \to 0$ as $K \downarrow 0$, and that this ratio equals $b(-k, \sqrt{w(k)})$. Deduce, as in the proof of Theorem 15.6(i), that $w(k) < 2|k|$ for all k negative enough.

(c) For raw SVI, $w(k) = a + b\big(\rho(k-m) + \sqrt{(k-m)^2 + \varsigma^2}\big)$ with $b \ge 0$, $|\rho| < 1$, $\varsigma > 0$, show that $w(k)/k \to b(1+\rho)$ as $k \to \infty$ and $w(k)/|k| \to b(1-\rho)$ as $k \to -\infty$. Show that if $b(1+\rho) = 2$, then $d_+(k) \to 0$ and the normalized call price tends to $\frac{1}{2}$ as $k \to \infty$. Conclude that a raw SVI slice coming from a pricing measure with $\mathbb{Q}(S_T = 0) = 0$ has $b(1+|\rho|) < 2$.

Exercise 15.4 *(Small volatility of volatility)* [⋆⋆] In the setting of Theorem 15.9, let V take the values $m(1-\delta)$ and $m(1+\delta)$ with probability $\frac{1}{2}$ each, where $m > 0$ and $0 < \delta < 1$, and let $h_k(v) = b(k, \sqrt{v})$.

(a) Show that $\mathbb{E}[h_k(V)] = h_k(m) + \frac{1}{2}h_k''(m)m^2\delta^2 + O(\delta^4)$ as $\delta \to 0$, for fixed k, and deduce that

$$w(k) = m + \frac{m^2\delta^2}{2}\Big(\frac{k^2}{2m^2} - \frac{1}{8} - \frac{1}{2m}\Big) + O(\delta^4).$$

You may use the formula for h_k''/h_k' in the proof of Theorem 15.9.

(b) Describe the shape of this approximate smile, and compare the values of k at which it crosses the level m with the threshold of Theorem 15.9(iii).

(c) This is a computational part. For $m = 0.04$ (a volatility of 20% over $T = 1$), compute the exact smile $\sqrt{w(k)}$ for $k \in \{0, \pm 0.1, \pm 0.3, \pm 0.5\}$ and $\delta \in \{0.2, 0.5\}$ by inverting the mixture prices, and compare with the approximation. When is the approximation accurate?

Exercise 15.5 *(The leverage effect in the Heston model)* [⋆⋆] Under $\mathbb{P}$, let v follow the square-root equation of Theorem 15.11 driven by B, and let W be a Brownian motion with $[W,B]_t = \rho t$. Let $R_t = \int_0^t \sqrt{v_s}\,\mathrm{d}W_s$, the martingale part of the log return $\log(S_t/S_0)$ over $[0,t]$. You may use that $\mathbb{E}\int_0^T v_s\,\mathrm{d}s < \infty$, and that for progressively measurable H, G with $\mathbb{E}\int_0^t H_s^2\,\mathrm{d}s$ and $\mathbb{E}\int_0^t G_s^2\,\mathrm{d}s$ finite, $\mathbb{E}\big[\int_0^t H\,\mathrm{d}W \int_0^t G\,\mathrm{d}B\big] = \rho\,\mathbb{E}\int_0^t H_s G_s\,\mathrm{d}s$.

(a) By applying Itô's formula to $e^{\kappa u}v_u$, show that $v_u = \theta + (v_0 - \theta)e^{-\kappa u} + \xi\int_0^u e^{-\kappa(u-s)}\sqrt{v_s}\,\mathrm{d}B_s$.

(b) Show that for $t, h \ge 0$, $\mathrm{Cov}(R_t, v_{t+h}) = \rho\xi\int_0^t e^{-\kappa(t+h-s)}\mathbb{E}[v_s]\,\mathrm{d}s$, and compute it when $v_0 = \theta$.

(c) Interpret the sign, and compare with Theorem 15.1(iii) and the right panel of Figure 15.1. How does a negative correlation between returns and future variance show in the smile?

Exercise 15.6 *(Dupire's formula in implied variance)* [⋆⋆] Let the call prices be given by a total implied variance $w(k,T)$, twice continuously differentiable in k and continuously differentiable in T, and let $\gamma(k,T) = e^{rT}C(F_Te^k, T)/F_T = b(k, \sqrt{w(k,T)})$.

(a) Show that, in the variables (k, T), Dupire's equation Equation (15.7) becomes $\partial_T \gamma = \frac{1}{2}\sigma(T, F_T e^k)^2(\partial_{kk}\gamma - \partial_k \gamma)$: the rate and the dividend yield disappear.

(b) Using the identities in the proof of Theorem 15.6, show that $\partial_T \gamma = \psi\, \partial_T w$ and $\partial_{kk}\gamma - \partial_k \gamma = 2\psi g$, with $\psi = N'(d_+)/(2\sqrt{w})$, and deduce formula Equation (15.11).

(c) Suppose $w(k, T) = \int_0^T \vartheta(t)^2\, \mathrm{d}t$ does not depend on k, for a continuous positive function ϑ. What is the local volatility, and what is the model?

(d) Explain why Equation (15.11) produces a negative local variance on part of the slice of Figure 15.2 with Vogt's parameters, as soon as the total variance increases with T there.

Exercise 15.7 *(The Heston characteristic function)* [⋆ ⋆ ⋆] Consider the Heston model of Theorem 15.11 under $\mathbb{Q}$, with $X_t = \log(S_t/F_t)$, so that $\mathrm{d}X_t = -\frac{1}{2}v_t\, \mathrm{d}t + \sqrt{v_t}\, \mathrm{d}\widetilde{W}_t$. Fix a real u and $T > 0$. In this exercise you may treat complex logarithms and square roots formally.

(a) Let $f(t, x, v)$ be complex valued and continuously differentiable once in t and twice in (x, v). Show that $f(t, X_t, v_t)$ has zero drift if and only if, along the path,

$$\partial_t f - \frac{v}{2}\,\partial_x f + \frac{v}{2}\,\partial_{xx} f + \rho\xi v\,\partial_{xv} f + \kappa(\theta - v)\,\partial_v f + \frac{\xi^2 v}{2}\,\partial_{vv} f = 0.$$

Which equation does the price $c(t, s, v)$ of a call satisfy, if $e^{-rt}c(t, S_t, v_t)$ is a martingale and c is smooth?

(b) Look for $f = \exp\left(iux + A(T-t) + D(T-t)v\right)$ with $A(0) = D(0) = 0$. Show that the equation of (a) holds if and only if $A' = \kappa\theta D$ and

$$D' = \frac{\xi^2}{2}D^2 - (\kappa - \rho\xi iu)D - \frac{1}{2}(u^2 + iu).$$

(c) Let $D_\pm = (\beta \pm d)/\xi^2$ be the roots of the right side, with β and d as in Equation (15.12). By computing the derivative of $\log\left((D - D_-)/(D - D_+)\right)$, derive the formula for D in Equation (15.12), and check that A given there satisfies $A' = \kappa\theta D$.

(d) Let $\xi \to 0$ (the variance becomes deterministic). Solve the equation for D in that case, and check that $\exp(A + Dv_0)$ is the characteristic function of $\mathcal{N}(-V/2, V)$ with $V = \int_0^T v_t\, \mathrm{d}t$ and $v_t = \theta + (v_0 - \theta)e^{-\kappa t}$.

Exercise 15.8 *(Fitting and checking a smile)* [⋆⋆] This is a computational exercise. Take the Heston model of Figure 15.3 with $\rho = -0.7$: $S_0 = 100$, $r = 2\%$, $q = 0$, $v_0 = \theta = 0.04$, $\kappa = 2$, $\xi = 0.5$, and $T = 0.5$. You may compute Heston call prices with Theorem 15.10 and Theorem 15.11, or with the function `heston_call` of the course code (`code/src/ieor222/volatility.py`).

(a) Compute the total implied variances w_i at $k_i = -0.4, -0.35, \ldots, 0.3$.

(b) Fit a raw SVI slice to the w_i by least squares, with $b \geq 0$, $|\rho| < 1$ and $\varsigma > 0$ (for instance with the function `least_squares` of `scipy.optimize`). Report the parameters and the largest error in volatility points.

(c) Check the fitted slice for butterfly arbitrage on $k \in [-2, 2]$, and its wings against Exercise 15.3.

(d) Price the digital call with strike F_T from the fitted smile with Theorem 15.12, and compare with the Heston price $e^{-rT}\mathbb{Q}(S_T > F_T) = -\partial_K C$, computed by a finite difference of Heston prices, and with the Black–Scholes digital at the at-the-money implied volatility.

Chapter 16

Numerical methods

Exercise 16.1 *(When confidence intervals lie)* [⋆] This is a computational exercise. Take $S_0 = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 1$, and the digital call that pays $\mathbf{1}_{\{S_T > K\}}$ at T, whose price is $p = e^{-rT}q$ with $q = \mathbb{Q}(S_T > K) = N(d_-)$ (Theorem 6.13). Let $Y = e^{-rT}\mathbf{1}_{\{S_T > K\}}$ and use the interval Equation (16.1) with $z = 1.96$.

(a) Show that $\mathrm{Var}(Y) = e^{-2rT}q(1-q)$, and that the half-width of the interval divided by p is approximately $1.96\sqrt{(1-q)/(qM)}$. How many paths make it 1% for $K = 100$, and for $K = 200$?

(b) Take $K = 200$. With probability $(1-q)^M$ all M payoffs vanish. What is the interval then, and does it contain p? Deduce an upper bound for the probability that the interval contains p, for $M = 10^2$, 10^3 and 10^4.

(c) Estimate this probability, the *coverage*, for $K \in \{100, 200\}$ and $M \in \{10^2, 10^3, 10^4, 10^5\}$, by repeating the whole experiment many times. *Hint:* the estimator and the sample variance depend on the payoffs only through the number of paths that end above K, whose law you know, so there is no need to simulate the paths. Compare with (b) and with 95%, and explain why this does not contradict Theorem 16.1.

Exercise 16.2 *(The explicit scheme as a tree)* [⋆] Let $r \geq 0$, and consider the explicit scheme for the Black–Scholes equation in log-price Equation (16.3), on a grid with $h|b| \leq 2a$, $k(2a/h^2 + c) \leq 1$ and $rk < 1$.

(a) Write the scheme as $U_j^{n+1} = (1-rk)(\pi_+ U_{j+1}^n + \pi_0 U_j^n + \pi_- U_{j-1}^n)$, with the probabilities of Theorem 16.9 written out in terms of a, b, r, h and k.

(b) Let Δ be a random variable equal to h, 0, $-h$ with probabilities π_+, π_0, π_-: the move of the log-price in one period of the tree. Compute $\mathbb{E}[\Delta]$ and $\mathbb{E}[\Delta^2]$, and compare them with the mean and the second moment of $\log S_{t+k} - \log S_t$ in the Black–Scholes model. Show that both differences are of order k^2, whatever the choice of h. Compare also the discount factor $1 - rk$ with e^{-rk}.

(c) Matching two moments is not enough. Compare $\mathbb{E}[\Delta^4]$ with the fourth moment of a normal variable with variance $\sigma^2 k$, and explain why the scheme converges only if h tends to zero

as well as k. What does the tree look like when h is fixed and $k \to 0$?

Exercise 16.3 *(A magic ratio)* [⋆⋆] Consider the heat equation $\partial_\tau u = a\,\partial_{yy} u$ and the explicit scheme, with $\lambda = ak/h^2$, and let u be a solution whose partial derivatives of order at most 6 are continuous and bounded.

(a) Show that the truncation error is

$$T_j^n = \frac{ah^2}{2}\Big(\lambda - \frac{1}{6}\Big)\partial_y^4 u(\tau_n, y_j) + O(k^2 + h^4).$$

(b) Deduce that for $\lambda = \frac{1}{6}$ the error of the scheme is $O(h^4)$, uniformly on the grid, when the boundary values are exact. You may use the argument of Theorem 16.10(iii)–(iv).

(c) This is a computational part. Take $a = \frac{1}{2}$ on $[0, \pi]$ with the exact solution $u(\tau, y) = e^{-a\tau}\sin y$ and zero boundary values. For $J = 10, 20, 40, 80$ intervals, $h = \pi/J$, $k = \lambda h^2/a$, run $N = \lceil 1/k \rceil$ steps and report the largest error at $\tau = Nk$, for $\lambda = 0.4$ and $\lambda = \frac{1}{6}$, with the observed orders of convergence.

(d) Which of the trinomial trees of Theorem 4.15 corresponds to $\lambda = \frac{1}{6}$? (Use Theorem 16.9.)

Exercise 16.4 *(Crank–Nicolson and positivity)* [⋆⋆] Consider the heat equation $\partial_\tau u = a\,\partial_{yy} u$ and the Crank–Nicolson scheme with $\lambda = ak/h^2$, written as $BU^{n+1} = CU^n$ with $B = I - \frac{k}{2}A$ and $C = I + \frac{k}{2}A$.

(a) On an interval with zero boundary values, show that if $\lambda \le 1$, then $U^n \ge 0$ implies $U^{n+1} \ge 0$. *Hint:* look at the weights of C, and show that $Bz \ge 0$ implies $z \ge 0$, as in the proof of Theorem 16.13(iii).

(b) Now work on the infinite grid $h\mathbb{Z}$, with bounded sequences. Show that $C = 2I - B$, so that one step maps U to $2B^{-1}U - U$. Find $\rho \in (0,1)$ and $C_\lambda > 0$ such that $z_j = C_\lambda \rho^{|j|}$ solves $Bz = e$, where $e_0 = 1$ and $e_j = 0$ for $j \neq 0$ (you may admit that it is the only bounded solution). Deduce that the spike e is mapped to a sequence whose value at 0 is $2/\sqrt{1 + 2\lambda} - 1$.

(c) Conclude that, on the infinite grid, the step maps every nonnegative bounded sequence to a nonnegative one if and only if $\lambda \le \frac{3}{2}$.

(d) The second differences of the call payoff on a grid whose nodes include $\log K$ vanish except at $\log K$: they form a spike. Since the scheme commutes with differences, use (b) to explain the sign of the Crank–Nicolson gamma at the strike after one step, for $\lambda \approx 22$ as in Figure 16.3, and the oscillations that follow.

Exercise 16.5 *(Richardson extrapolation of a biased estimator)* [⋆⋆] Work in the setting of Theorem 16.4: $Y_h = e^{-rT}(S_T^h)^2$, with the Euler scheme on S and $n = T/h$ steps, and $p = S_0^2 e^{(r+\sigma^2)T}$.

(a) From the closed form of $\mathbb{E}[Y_h]$, show that the bias has an expansion $\beta(h) = \beta_1 h + \beta_2 h^2 + O(h^3)$, and compute β_1 and β_2.

(b) Let $Z = 2\hat{p}_{M_1,h/2} - \hat{p}_{M_2,h}$, where the two Monte Carlo estimators, with steps $h/2$ and h, use independent samples. Show that $\mathbb{E}[Z] - p = -\frac{1}{2}\beta_2 h^2 + O(h^3)$, and compute $\mathrm{Var}(Z)$.

(c) Show that, for small ε, a root mean square error ε can be reached with Z at a cost of order $\varepsilon^{-5/2}$, against ε^{-3} in Theorem 16.3.

(d) Take $S_0 = 1$, $r = 5\%$, $\sigma = 40\%$, $T = 1$ and $\varepsilon = 10^{-3}$. Using the exact bias and variance (the functions of `code/src/ieor222/numerics.py` compute them), find the cheapest choice of n and M for the plain estimator, and of n, M_1 and M_2 for Z, and compare the costs, counted in normal draws. Compare also with the choice of Theorem 16.3(i) with $c_2 = 0.039$ and $v_2 = 1.37$.

Exercise 16.6 *(An up-and-out call on a grid)* [⋆⋆] This is a computational exercise. Take $S_0 = K = 100$, a barrier $B = 120$, $r = 5\%$, $\sigma = 20\%$ and $T = 1$; the up-and-out call costs 1.1761 (Theorem 13.5). As in Theorem 13.6, accept that while the option is alive its price is $v(t, S_t)$, where v solves the Black–Scholes equation on $[0, T) \times (0, B)$ with $v(t, B) = 0$ and $v(T, x) = (x - K)^+$ for $x < B$.

(a) Write the problem in log-price on a grid whose top node is $\log B$ and on which $\log S_0$ is a node: $h = \log(B/S_0)/m_0$ for an integer m_0, extending at least $6\sigma\sqrt{T}$ below $\log S_0$. What should the lower boundary value be?

(b) Solve it by Crank–Nicolson with $k \approx h/\sigma$, with and without Rannacher's start, for $m_0 = 5, 10, \dots, 160$, and report the errors at S_0 and the observed orders. Where is the payoff not smooth, and why does Rannacher's start matter even though the payoff is continuous at the strike?

(c) Now let the barrier be monitored only at the dates iT/m, $1 \le i \le m$. Solve the equation between two dates on a grid that extends above B (with the boundary values of a call at the top), and at each monitoring date set the solution to 0 above B. Compute the price for $m = 4, 12, 52, 250$. Compare the convergence in m_0 when the value at the node $\log B$ itself is set to 0, and when it is halved (the average of the solution over the cell around that node). Compare the prices with the continuously monitored price with the barrier shifted to $Be^{\beta\sigma\sqrt{T/m}}$, $\beta \approx 0.5826$ (Theorem 13.7).

Exercise 16.7 *(The exercise region of the American put)* [⋆⋆⋆] Let W be a Brownian motion with its natural filtration, $K > 0$, $\sigma > 0$, $r \ge 0$, and $R_s = \exp\left(\sigma W_s + (r - \frac{\sigma^2}{2})s\right)$. For $\theta > 0$, let $\mathcal{T}_\theta$ be the set of stopping times with values in $[0, \theta]$, and

$$V(\theta, x) = \sup_{\varrho \in \mathcal{T}_\theta} \mathbb{E}\big[e^{-r\varrho}(K - xR_\varrho)^+\big], \qquad x > 0,$$

so that the price Equation (16.7) is $v(t, x) = V(T - t, x)$, by the restart property of Brownian motion. You may use optional sampling for continuous martingales and bounded stopping times: if $\varrho \le \varrho'$ are in $\mathcal{T}_\theta$, then $\mathbb{E}[e^{-r\varrho'}R_{\varrho'} \mid \mathcal{F}_\varrho] = e^{-r\varrho}R_\varrho$. Let $E_\theta = \{x > 0 : V(\theta, x) = (K - x)^+\}$ be the exercise region with time θ to maturity.

(a) Show that $x \mapsto V(\theta, x)$ is convex and nonincreasing, and that $V(\theta, y) \le V(\theta, x) + x - y$ for $0 < y < x$.

(b) Show that if $x \in E_\theta$ and $0 < y < x$, then $y \in E_\theta$; that E_θ is closed in $(0, \infty)$; and that $E_\theta \cap [K, \infty) = \emptyset$. Deduce that E_θ is empty or of the form $(0, b]$ with $b < K$.

(c) Let $r > 0$ and $L^* = 2rK/(2r + \sigma^2)$. Using Theorem 13.12, show that $(0, L^*] \subseteq E_\theta$, so that $E_\theta = (0, b(\theta)]$ with $L^* \leq b(\theta) < K$.

(d) Show that $\theta \mapsto V(\theta, x)$ is nondecreasing, and deduce that b is nonincreasing in θ: in calendar time $t = T - \theta$, the exercise boundary rises towards maturity.

(e) Let $r = 0$. Show that $V(\theta, x) = \mathbb{E}[(K - xR_\theta)^+] > (K - x)^+$ for every $x > 0$, so that $E_\theta = \emptyset$: the American put is never exercised early.

Part V

Extras

Chapter 17

Arbitrage-free option surfaces

Unless an exercise says otherwise, prices are normalized as in Equation (17.1), so that the stock and the bond both cost 1, and the market is frictionless.

Exercise 17.1 *(A calendar spread)* [⋆] Take $r = q = 0$, so that $F_T = S_0 = 1$ and $Z_T = S_T$. Two calls with the same strike $x = 1$ and maturities $T_1 < T_2$ are quoted at $c(T_1, 1) = 0.060$ and $c(T_2, 1) = 0.055$.

(a) Buy the T_2 call and sell the T_1 call. At T_1, if $S_{T_1} > 1$, the call you sold is exercised: deliver a borrowed share against the strike, and keep the short position until T_2. Show that this strategy collects 0.005 at time 0 and never has a negative value at T_2, whatever the path of S.

(b) Now let r, q be arbitrary constants, $x > 0$, and suppose that the normalized prices of Equation (17.1) satisfy $c(T_2, x) < c(T_1, x)$. Buy e^{rT_2}/F_{T_2} calls with strike xF_{T_2} and maturity T_2 and sell e^{rT_1}/F_{T_1} calls with strike xF_{T_1} and maturity T_1. If the latter end in the money, borrow at T_1 the amount you pay, and sell e^{rT_2}/F_{T_2} forward contracts on the stock for delivery at T_2, at the forward price $S_{T_1}e^{(r-q)(T_2-T_1)}$ (which costs nothing). Show that this is an arbitrage. You may write every payoff in terms of $Z_t = S_t/F_t$.

(c) Show that the trade at T_1 is needed: with the quotes of (a), find a probability measure for the pair (S_{T_1}, S_{T_2}) under which every static portfolio of the two calls, the stock and the bond (held to T_2, the payoff of the T_1 call kept in cash) costs the expectation of its payoff, so that none is an arbitrage. *Hint:* take S_{T_1} and S_{T_2} independent, each with two values. Which property of the price process does your measure lack?

Exercise 17.2 *(Different grids at different maturities)* [⋆⋆] At maturity T_1 the only quotes are $c(T_1, 1) = 0.25$ and $c(T_1, 2) = 0$; at $T_2 > T_1$ they are $c(T_2, 0.5) = 0.6$, $c(T_2, 1.5) = 0.13$ and $c(T_2, 3) = 0$. As always, $c(T, 0) = 1$.

(a) Compute the linear interpolation $\hat{c}_1$ of the T_1 quotes (Theorem 17.2) at $x = 0.5$ and $x = 1.5$, and check that the condition of SANOS Theorem 2.6, $\hat{c}_1(x) \le c(T_2, x)$ at the T_2 strikes, fails.

(b) Show that the quotes are nevertheless free of static arbitrage: let Z_1 take the values 0.5 and

1.5 with probability $\frac{1}{2}$ each, find the law of Z_2 given by Theorem 17.6 for the T_2 quotes, and check the convex order with Theorem 17.4. Then find a martingale coupling explicitly, by splitting each value of Z_1 between neighbouring values of Z_2.

(c) Let $\ell(x) = \max(1 - x,\, 0.5 - 0.25x)$ for $0 \le x \le 1$, $\ell(x) = \max(1 - 0.75x,\, 0)$ for $1 \le x \le 2$, and $\ell(x) = 0$ for $x \ge 2$. Show that the call function of every law on $[0, \infty)$ with mean 1 that reproduces the T_1 quotes is at least ℓ, and that for each x some such law attains $\ell(x)$. Is ℓ itself the call function of such a law? Deduce a condition on the T_2 quotes that is necessary for the absence of static arbitrage, and compare it with the sufficient condition of (a).

Exercise 17.3 *(Smoothing raises prices)* [⋆] Let q put mass $\frac{1}{2}$ on each of the normalized strikes 0.9 and 1.1.

(a) Compute the linear call function $\sum_i q^i (K^i - x)^+$ at $x = 0.9, 1, 1.1$, and the total implied volatility of the price at $x = 1$ (use $s = 2N^{-1}((1 + c)/2)$ at the money, Theorem 15.4).

(b) Compute the SANOS price $\sum_i q^i \,\mathrm{Call}(K^i, 1, v)$ for $v = 0.0025$ and $v = 0.01$, and check Theorem 17.8(iii). Why is the inequality strict for $v > 0$?

(c) The call with $x = 1$ is quoted with ask 0.052. Find the largest v for which the SANOS price at $x = 1$ stays below the ask, and the SANOS prices at $x = 0.9$ and $x = 1.1$ for that v. (This is the mechanism of SANOS Proposition 3.8.)

Exercise 17.4 *(Discrete local volatility on three strikes)* [⋆⋆] Take the grid $K^1 = 1 - a < K^2 = 1 < K^3 = 1 + b$ with $a \in (0, 1)$, $b > 0$, one discrete local volatility $\Sigma > 0$ at K^2, a step $\Delta > 0$, and the weights Equation (17.4).

(a) Write the 3×3 matrix $\mathbf{M}$ and compute $\mathbf{Q} = \mathbf{M}^{-1}$. Check that $\mathbf{Q}$ is a martingale transition matrix and that the boundary strikes are absorbing.

(b) Starting from the unit mass at 1, compute the law $q_1 = \mathbf{Q}e_2$, its mean and its variance. Show that the variance is $\Sigma^2\Delta/(1 + w^{2-} + w^{2+})$, and discuss the limits $\Delta \to 0$ and $\Delta \to \infty$.

(c) Check that Equation (17.5) applied to q_1 returns Σ. Give the numbers for $a = b = 0.1$, $\Sigma = 0.2$ and $\Delta = 1$.

Exercise 17.5 *(Dupire's formula on a grid)* [⋆⋆] Let the strikes be equally spaced, $K^i = K^0 + ih$, and the expiries $T_j = j\Delta$.

(a) Show that $\mathbf{M}_j q_j = q_{j-1}$ (Theorem 17.10) reads, at the inner strikes and with $a^i = (\Sigma^i_j K^i)^2$,

$$\frac{q^i_j - q^i_{j-1}}{\Delta} = \frac{1}{2h^2}\Big(a^{i+1}q^{i+1}_j - 2a^i q^i_j + a^{i-1}q^{i-1}_j\Big),$$

the implicit Euler scheme for $\partial_T p = \frac{1}{2}\partial_{xx}(\sigma^2 x^2 p)$, the forward equation Equation (15.9) in normalized units, if $q^i_j \approx p(T_j, K^i)h$.

(b) Let $c(T, x)$ be a call-price surface that is twice continuously differentiable in T and four times in x, with $\partial_{xx}c > 0$, and let $c^i_j = c(T_j, K^i)$. Show that Σ^i_j of Equation (17.5) satisfies $(\Sigma^i_j)^2 = \sigma_D(T_j, K^i)^2 + O(\Delta + h^2)$ locally uniformly, where $\sigma^2_D = 2\partial_T c/(x^2\partial_{xx}c)$

is Dupire's local variance in normalized units (Equation (15.8) with $r = q = 0$). Use $\Gamma^i_j = (c^{i+1}_j - 2c^i_j + c^{i-1}_j)/h^2$ on this grid.

(c) In the notes, the largest errors were 2.7, 0.6 and 0.15 volatility points for $(\Delta, h) = (1/12, 0.05)$, $(1/52, 0.02)$ and $(1/252, 0.01)$ (with a uniform grid in $\log x$). Which of the two error terms dominates?

Exercise 17.6 *(Linear programming duality and arbitrage with spreads)* [⋆⋆⋆] One maturity, no smoothing. Let $G = \{0 = g^0 < g^1 < \cdots < g^m\}$ be the possible values of Z_T, and let calls with strikes $x_1, \dots, x_L \in G$ be quoted with bids b_ℓ and asks $a_\ell \geq b_\ell$. You may use Farkas' lemma in the following form: for matrices A, E and vectors h, e, exactly one of the two systems

$$\text{(P)}\quad q \geq 0,\ Aq \leq h,\ Eq = e; \qquad\qquad \text{(D)}\quad y \geq 0,\ \lambda \text{ free},\ A^\top y + E^\top \lambda \geq 0,\ h^\top y + e^\top \lambda < 0$$

has a solution.

(a) Write the existence of a probability vector q on G with mean 1 and $b_\ell \leq \sum_i q^i (g^i - x_\ell)^+ \leq a_\ell$ for every ℓ as a system (P).

(b) Interpret a solution of (D) as a portfolio: λ_0 bonds, λ_1 shares, $\alpha_\ell \geq 0$ calls bought at the ask and $\beta_\ell \geq 0$ calls sold at the bid. Deduce: either the quotes are consistent with some law on G, or there is a portfolio whose payoff is nonnegative at every point of G and whose cost is negative.

(c) Show that a payoff of this form that is nonnegative at the points of G is nonnegative on $[0, g^m]$, but give an example in which it is negative for some $z > g^m$. What does this say about the choice of the boundary strike K^N in SANOS (Remark 2.7 of the paper chooses it well beyond the largest quoted strike)?

Exercise 17.7 *(The conditional test for Gaussian returns)* [⋆⋆] In the setting of Theorem 17.13, suppose that given $\mathcal{G}$ the vector R is Gaussian with mean μ and covariance Γ, where μ and Γ are bounded $\mathcal{G}$-measurable and Γ is positive definite.

(a) Show that $\mathbb{E}\big[\exp(-a^\top R + \tfrac{1}{2}|a|^2) \mid \mathcal{G}\big] = \exp\left(-a^\top \mu + \tfrac{1}{2} a^\top (I + \Gamma) a\right)$ for bounded $\mathcal{G}$-measurable a, and deduce that $a^* = (I + \Gamma)^{-1}\mu$.

(b) Compute $\mathbb{E}[a^{*\top} R]$ and $\mathbb{E}|a^*|^2$, and check the inequality of Theorem 17.13 directly.

(c) The conditional Sharpe ratio $a^\top \mu / \sqrt{a^\top \Gamma a}$ of a portfolio a is maximized by $a = \Gamma^{-1}\mu$. When are a^* and $\Gamma^{-1}\mu$ proportional? For daily returns Γ is small compared with I (standard deviations of a few percent): what is a^* then, approximately, and what does $\mathbb{E}[a^{*\top} R]$ measure?

(d) The test of the notes sums the captured returns over D dates. If each date has the same small drift and the captured returns of different dates are uncorrelated, how does the expected test statistic grow with D?

Exercise 17.8 *(Calibrating SANOS)* [⋆⋆] This is a computational exercise. Use the functions `sanos_example` and `sanos_calibrate` of the course code (`code/src/ieor222/surfaces.py`), which build the running example of the notes and solve the linear program of Theorem 17.9.

(a) Fit the running example with $\eta = 0$, 0.1, 0.25, 0.5 and 0.9. For each, report the fraction of fitted prices inside their spreads and the largest error in implied volatility against the Heston prices.

(b) Explain the results with Theorem 17.8(iii): why does a large η make the fit fail, and at which strikes and expiries do you expect the misses?

(c) Solve the program with the spreads as hard constraints (`hard=True`) for $\eta = 0.25$. Is it feasible? What would infeasibility tell you, by Theorem 17.9(iii)?

(d) With $\eta = 0$ the fit is also inside the spreads. Plot its implied volatility at one expiry between two quotes, and explain why SANOS recommends $\eta > 0$ all the same.

Chapter 18

Hawkes processes and limit order books

Exercise 18.1 *(The exponential kernel)* [⋆] Let $d = 1$, $\mu > 0$ and $\varphi(t) = \alpha\beta e^{-\beta t}$ with $0 \le \alpha < 1$, $\beta > 0$, and let $\Lambda = \mu/(1-\alpha)$.

(a) Using the renewal equation of Theorem 18.6, show that $m(t) = \mathbb{E}[\lambda(t)]$ is differentiable, with $m' = \beta(\mu - (1-\alpha)m)$ and $m(0) = \mu$, and deduce $m(t) = \Lambda - (\Lambda - \mu)e^{-\beta(1-\alpha)t}$.

(b) Compute $\mathbb{E}[N_t]$ and the limit of $\mathbb{E}[N_t] - \Lambda t$ as $t \to \infty$.

(c) With time in seconds and $\beta = 10$, how long does it take for $m(t)$ to come within 1% of Λ when $\alpha = 0.9$? When $\alpha = 0.99$? Comment in the light of Theorem 18.5.

Exercise 18.2 *(The size of a cluster)* [⋆⋆] Let $d = 1$ with branching ratio $a = \|\varphi\|_1 \in (0,1)$, and let S be the number of events in the cluster of one event, itself included (Theorem 18.7). Let $e = \mathbb{E}[S] = 1/(1-a)$.

(a) Show that $\mathbb{E}[S^2] < \infty$ (consider the clusters truncated after n generations) and that $\mathrm{Var}(S) = a/(1-a)^3$. *Hint:* the root has a Poisson(a) number of children, each of which starts an independent copy of the cluster.

(b) You may use the *hitting time theorem*: for a Galton–Watson tree whose offspring numbers are i.i.d. copies of K, $\mathbb{P}(S = n) = \frac{1}{n}\mathbb{P}(K_1 + \cdots + K_n = n-1)$ for $n \ge 1$, with $K_1, K_2, \dots$ i.i.d. copies of K. Deduce that S has the Borel law $\mathbb{P}(S = n) = e^{-an}(an)^{n-1}/n!$.

(c) Show that the expected number of immigrants divided by the expected number of events in a cluster is $1 - a$, and that the expected number of generation-g events divided by the expected size of a cluster is $(1-a)a^g$. What are the exogenous fraction of the activity and the mean generation of an event when $a = 0.9$?

Exercise 18.3 *(Thinning with a delayed kernel)* [⋆⋆] Let $d = 1$, $\mu > 0$ and $\varphi(t) = \alpha\beta^2 t e^{-\beta t}$ with $\alpha \in (0,1)$ and $\beta > 0$: an event excites most strongly after a delay.

(a) Show that $\|\varphi\|_1 = \alpha$ and that φ attains its maximum $\alpha\beta/e$ at $t = 1/\beta$. Explain why the

hypothesis of Theorem 18.8 fails, and what would go wrong if the algorithm were run with $M = \lambda(s+)$ anyway.

(b) Let $\tilde{\varphi}(x) = \sup_{v \geq x} \varphi(v)$. Show that $\tilde{\varphi}(x) = \varphi(\max(x, 1/\beta))$, and that for a history h with events $s_1 < \cdots < s_k$ the function $M(s) = \mu + \sum_l \tilde{\varphi}(s - s_l)$ is nonincreasing and satisfies $M(s) \geq \lambda(u \mid h)$ for all $u \geq s > s_k$.

(c) Show that the proof of Theorem 18.8 goes through when, in step 1, $M = M(s)$ is any bound with $M(s) \geq \bar{\lambda}(u \mid h)$ for all $u > s$ and $\sup_{s > s_k} M(s) < \infty$. Write the modified algorithm for this kernel.

Exercise 18.4 *(The fitted compensator matches the count)* [⋆⋆] Fix the decays β_{ij} of a d-variate Hawkes process with exponential kernels observed on $[0, T]$. By Theorem 18.9, for each type i the log-likelihood has the form $\ell^i(\theta) = \sum_{k:\, \kappa_k = i} \log(\theta^\top x_k) - \theta^\top c$ in $\theta = (\mu_i, \alpha_{i1}, \ldots, \alpha_{id})$, with vectors $x_k, c \in [0, \infty)^{d+1}$ that do not depend on θ, and $\theta^\top c = \int_0^T \lambda^i$.

(a) Let $N_T^i \geq 1$ and let $\hat{\theta} \in [0, \infty)^{d+1}$ maximize ℓ^i, with $\hat{\theta}^\top x_k > 0$ for all k. By looking at $t \mapsto \ell^i(t\hat{\theta})$, $t > 0$, show that the fitted compensator satisfies $\int_0^T \hat{\lambda}^i(s)\, \mathrm{d}s = N_T^i$.

(b) Deduce the maximum likelihood estimator of the rate of a Poisson process. For the transformed gaps $\hat{\tau}_k$ of Theorem 18.10 under the fitted model ($d = 1$), show that $\sum_{k \leq N_T} \hat{\tau}_k \leq N_T$, with a difference equal to the compensator of the last incomplete gap. Why does a goodness-of-fit test have to look at the shape of the distribution of the $\hat{\tau}_k$ rather than at their mean?

(c) For $d = 1$ and $n = N_T \geq 2$, show that ℓ^1 is strictly concave on $(0, \infty)^2$, so that its maximizer there, if it exists, is unique.

Exercise 18.5 *(Volatility from order flow)* [⋆⋆] Let (N^+, N^-) count upward and downward ticks of a price $P = N^+ - N^-$, and let it be a bivariate Hawkes process with baseline (ν, ν), $\nu > 0$, self-excitation $\varphi_{++} = \varphi_{--} = \varphi_s$ and cross-excitation $\varphi_{+-} = \varphi_{-+} = \varphi_c$, with $g_s = \|\varphi_s\|_1$, $g_c = \|\varphi_c\|_1$ and $\int_0^\infty t^{1/2}(\varphi_s + \varphi_c)(t)\, \mathrm{d}t < \infty$.

(a) Find the eigenvalues and eigenvectors of G, show that $\varrho(G) < 1$ if and only if $g_s + g_c < 1$, and compute Λ.

(b) Assume $g_s + g_c < 1$. Using Theorem 18.11, show that $P_{Tv}/\sqrt{T}$ converges in law to σW_v for a Brownian motion W, with $\sigma^2 = 2\Lambda_+/(1 - g_s + g_c)^2$, $\Lambda_+ = \nu/(1 - g_s - g_c)$.

(c) Compare σ^2 with its value 2ν without excitation, first for pure self-excitation ($g_c = 0$), then for pure cross-excitation ($g_s = 0$, $g_c = x$). For which x is the price less volatile than without excitation? Interpret cross-excitation as mean reversion of the price.

Exercise 18.6 *(The CIR limit for the exponential kernel)* [⋆⋆⋆] In the setting of Theorem 18.12, let $\varphi^T(t) = a_T \beta e^{-\beta t}$ with $a_T = 1 - \lambda/T$ for $T > \lambda > 0$, $\mu > 0$, $m = 1/\beta$, and $C_t^T = (1 - a_T)\lambda_{Tt}^T$.

(a) Explain why C^T is a Markov process with generator

$$L^T f(c) = -T\beta(c - (1 - a_T)\mu) f'(c) + \frac{Tc}{1 - a_T}\Big(f(c + (1 - a_T)a_T\beta) - f(c)\Big),$$

and show that for every f with bounded continuous derivatives up to order 3 and every $K > 0$, $\sup_{0\le c\le K} |L^T f(c) - Lf(c)| \to 0$ as $T \to \infty$, where $Lf(c) = \frac{\lambda}{m}(\mu - c)f'(c) + \frac{\lambda}{2m^2} c\, f''(c)$ is the generator of the CIR process of Theorem 18.13(i).

(b) For a fixed branching ratio $a < 1$, you may use that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[f(\lambda_t)] = \mathbb{E}[\mathcal{L}f(\lambda_t)]$ for polynomials f, where $\mathcal{L}f(x) = -\beta(x-\mu)f'(x) + x\big(f(x + a\beta) - f(x)\big)$ is the generator of the intensity. Write the linear differential equations solved by $\mathbb{E}[\lambda_t]$ and $\mathbb{E}[\lambda_t^2]$ for $t \ge 0$ (they give the exact first two moments of C_1^T), find their limits as $t \to \infty$, and deduce that the stationary mean and variance of C^T are μ and $\mu a_T^2 \beta/2$.

(c) Show that the Gamma law with shape $2\mu m$ and scale $1/(2m)$ is stationary for the CIR limit, in the sense that its density p satisfies $\frac{1}{2}(\sigma^2 x p)'' - (\kappa(\mu - x)p)' = 0$ with $\kappa = \lambda/m$ and $\sigma^2 = \lambda/m^2$, and compare its mean and variance with (b). When does the Feller condition of Theorem 14.9 hold?

(d) Let κ, σ be as in (c), $u \ge 0$, $t > 0$, and let $A, Q \ge 0$ solve $A' = \kappa\mu Q$ and $Q' = -\kappa Q - \frac{1}{2}\sigma^2 Q^2$ on $[0, t]$, with $A(0) = 0$, $Q(0) = u$. Find A and Q. For the CIR limit X with $X_0 = 0$, show that $M_s = \exp(-A(t-s) - Q(t-s)X_s)$, $s \le t$, is a martingale, and deduce $\mathbb{E}[e^{-uX_t}] = \big(1 + u\sigma^2(1 - e^{-\kappa t})/(2\kappa)\big)^{-2\kappa\mu/\sigma^2}$. Conclude that X_1 has the Gamma law with shape $2\mu m$ and scale $(1 - e^{-\lambda/m})/(2m)$. You may use that $X \ge 0$, that $\mathbb{E}\int_0^t X_s\,\mathrm{d}s < \infty$, and that Laplace transforms on $[0, \infty)$ determine laws.

Exercise 18.7 *(Mittag-Leffler kernels)* [⋆ ⋆ ⋆] Let $0 < \alpha < 1$, $\lambda > 0$, $k(t) = \lambda t^{\alpha-1}/\Gamma(\alpha)$, and let $f = f^{\alpha,\lambda}$ and $F = F^{\alpha,\lambda}$ be as in Theorem 18.13. You may use the Beta integral $\int_0^t (t-s)^{p-1} s^{q-1}\,\mathrm{d}s = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} t^{p+q-1}$ for $p, q > 0$.

(a) Show that $f(t) = \sum_{n\ge 0} (-1)^n \lambda^{n+1} t^{\alpha(n+1)-1}/\Gamma(\alpha(n+1))$, that $f(t) \sim k(t)$ as $t \downarrow 0$, and that $f = k - k * f$: f is the *resolvent* of k. Justify the termwise operations.

(b) Show that $F(t) = \int_0^t f = 1 - E_\alpha(-\lambda t^\alpha)$ and that $F = k * (1 - F)$.

(c) Let $\sigma \ge 0$ and let $Y \ge 0$ be a continuous adapted process with $Y_t = F(t) + \sigma \int_0^t f(t-s)\sqrt{Y_s}\,\mathrm{d}B_s$ for all t. Assuming that stochastic and ordinary integrals may be exchanged (a stochastic Fubini theorem), show that $Y_t = \int_0^t k(t-s)(1 - Y_s)\,\mathrm{d}s + \sigma \int_0^t k(t-s)\sqrt{Y_s}\,\mathrm{d}B_s$: the two forms of the rough CIR equation in Section 18.6 agree.

Exercise 18.8 *(Estimating the branching ratio from counts)* [⋆⋆] This is a computational exercise. Let $d = 1$ and $\varphi(t) = a\beta e^{-\beta t}$ with $a \in (0, 1)$.

(a) Assume that, for a window of length w much longer than the memory of the process, the count N_w of a stationary process has the mean and variance that Theorem 18.11 suggests. Show that $\mathrm{Var}(N_w)/\mathbb{E}[N_w] \approx (1 - a)^{-2}$, and propose an estimator $\hat{a}$ of a from the counts $n_1, \dots, n_K$ of a path in K disjoint windows of length w.

(b) Simulate a path with $(\mu, a, \beta) = (1, 0.8, 1)$ on $[0, 2 \times 10^5]$ (with the code of the chapter or your own). Compute $\hat{a}$ for $w = 1, 10, 100, 1000$, and the maximum likelihood estimate of a on the same path.

(c) Explain the values you find for small and for large w, in terms of the time scale $1/(\beta(1-a))$ of Theorem 18.5 and of the number of windows. Why might one still prefer $\hat{a}$ to maximum

likelihood on real data?

Chapter 19

Rough volatility

Exercise 19.1 *(Fractional Gaussian noise)* [⋆] Let $G_n = W^H_{n+1} - W^H_n$, $n \in \mathbb{N}$, and extend ρ_H to $\mathbb{Z}$ by $\rho_H(-k) = \rho_H(k)$.

(a) Show that $\operatorname{Cov}(G_j, G_l) = \rho_H(l-j)$ for all $j, l \in \mathbb{N}$.

(b) Show that $\sum_{k=-m}^{m} \rho_H(k) = (m+1)^{2H} - m^{2H}$ for every $m \in \mathbb{N}$. Deduce that $\sum_{k\in\mathbb{Z}} \rho_H(k) = 0$ if $H < \frac{1}{2}$ (the series converges absolutely), and that the partial sums tend to $+\infty$ if $H > \frac{1}{2}$.

(c) Compute $\operatorname{Var}(G_0 + \cdots + G_{n-1})$ in two ways, and compare its growth in n with that of a sum of n independent standard normal variables.

(d) With $H \approx 0.1$, are the daily increments of log-volatility in the RFSV model long memory in the sense $\sum_n |\rho_H(n)| = \infty$?

Exercise 19.2 *(Quadratic variation of fBM)* [⋆⋆] Let $t > 0$, $t_j = jt/n$ and $V_n = \sum_{j=0}^{n-1} \left(W^H_{t_{j+1}} - W^H_{t_j}\right)^2$. You may use that $\operatorname{Cov}(X^2, Y^2) = 2\operatorname{Cov}(X,Y)^2$ for a centred Gaussian vector (X, Y).

(a) Show that $\mathbb{E}[V_n] = t^{2H} n^{1-2H}$.

(b) Show that $\operatorname{Var} V_n = 2(t/n)^{4H} \sum_{j,l=0}^{n-1} \rho_H(j-l)^2 \le 2(t/n)^{4H}\, n \sum_{k\in\mathbb{Z}} \rho_H(k)^2$, and that the last series converges when $H < \frac{3}{4}$.

(c) Deduce that $V_n \to \infty$ in probability if $H < \frac{1}{2}$, and $V_n \to 0$ in L^1 if $H > \frac{1}{2}$.

(d) You may use that for a continuous semimartingale X the sums $\sum_j (X_{t_{j+1}} - X_{t_j})^2$ along partitions of $[0,t]$ with mesh tending to 0 converge in probability to a finite limit (Revuz and Yor, *Continuous Martingales and Brownian Motion*, Chapter IV, Section 1). Show that W^H is not a semimartingale when $H < \frac{1}{2}$. Why does the argument say nothing when $H > \frac{1}{2}$?

Exercise 19.3 *(Consistency of the GJR statistic)* [⋆] Let $X = \nu W^H$ with $\nu > 0$ and $H < \frac{3}{4}$, let $\Delta > 0$, and $m_N(\Delta) = \frac{1}{N}\sum_{j=1}^{N} \left(X_{j\Delta} - X_{(j-1)\Delta}\right)^2$ (the statistic of Equation (19.4) with $q = 2$ and one starting point).

(a) Show that $\mathbb{E}[m_N(\Delta)] = \nu^2 \Delta^{2H}$ and, using the computation of Exercise 19.2(b), that $\operatorname{Var} m_N(\Delta) \le \frac{2\nu^4 \Delta^{4H}}{N} \sum_{k \in \mathbb{Z}} \rho_H(k)^2$.

(b) For $0 < \Delta_1 < \Delta_2$, show that $\hat{H}_N = \log\left(m_N(\Delta_2)/m_N(\Delta_1)\right)/\left(2\log(\Delta_2/\Delta_1)\right) \to H$ in probability as $N \to \infty$.

(c) For $H = 0.14$ and $N = 3500$ (fourteen years of daily data), bound the standard deviation of $m_N(1)/\nu^2$; you may evaluate $\sum_k \rho_H(k)^2$ numerically.

Exercise 19.4 *(Why realized variance looks smoother)* [⋆⋆] This exercise follows Appendix C of Gatheral, Jaisson and Rosenbaum (2018). Let $\delta > 0$, $A_t = \frac{1}{\delta} \int_t^{t+\delta} W_s^H \, \mathrm{d}s$ (an average over a window of length δ), and $D = A_{t+\Delta} - A_t$ for $\Delta > 0$.

(a) Show that $\mathbb{E}[D^2] = \frac{1}{2\delta^2} \int_0^\delta \int_0^\delta g(u - v) \, \mathrm{d}u \, \mathrm{d}v$ with $g(x) = |x + \Delta|^{2H} + |x - \Delta|^{2H} - 2|x|^{2H}$.

(b) Let $p = 2H + 2$ and $F(x) = |x|^p/(p(p-1))$, so that $F''(x) = |x|^{2H}$. Using $\int_0^\delta \int_0^\delta \varphi(u - v) \, \mathrm{d}u \, \mathrm{d}v = \int_{-\delta}^{\delta} (\delta - |x|)\varphi(x) \, \mathrm{d}x$ and $\int_{-\delta}^{\delta} (\delta - |x|) F''(c + x) \, \mathrm{d}x = F(c + \delta) - 2F(c) + F(c - \delta)$, show that $\mathbb{E}[D^2] = \Delta^{2H} f(\delta/\Delta)$ with

$$f(\vartheta) = \frac{(1 + \vartheta)^p + |1 - \vartheta|^p - 2 - 2\vartheta^p}{p(p-1)\vartheta^2}.$$

(c) Show that $f(\vartheta) = 1 - \frac{2\vartheta^{2H}}{p(p-1)} + O(\vartheta^2)$ as $\vartheta \downarrow 0$, and deduce that for δ/Δ small the slope $\frac{\mathrm{d} \log \mathbb{E}[D^2]}{\mathrm{d} \log \Delta}$ exceeds $2H$.

(d) (*Computational.*) For $H = 0.14$, fit $\log \mathbb{E}[D^2] = c + 2H_{\text{eff}} \log \Delta$ by least squares on $\Delta = 1, \dots, 30$ (days) for $\delta = \frac{1}{24}$ and $\delta = \frac{1}{3}$, then on $\Delta = 1, \dots, 100$. Compare with Theorem 19.8, Figure 19.2 and Table C.1 of Gatheral, Jaisson and Rosenbaum (0.161 and 0.184).

Exercise 19.5 *(Forward variance and the first-order skew)* [⋆⋆] Consider the rough Bergomi model of Theorem 19.9, with $\alpha = H - \frac{1}{2}$, and fix $u > 0$.

(a) Show that for $0 \le t \le u$,

$$\xi_t(u) := \mathbb{E}^{\mathbb{Q}}[v_u \mid \mathcal{F}_t] = \xi_0(u) \exp\Big(\eta \sqrt{2H} \int_0^t (u - s)^\alpha \, \mathrm{d}W_s - \frac{\eta^2}{2}\big(u^{2H} - (u - t)^{2H}\big)\Big).$$

(b) Show that $\mathrm{d}\xi_t(u) = \xi_t(u) \, \eta \sqrt{2H} (u - t)^\alpha \, \mathrm{d}W_t$ on $[0, u)$, so that $(\xi_t(u))_{t<u}$ is a martingale, and compute $\mathrm{d}[\log S, \xi_\cdot(u)]_t$.

(c) Let $\xi_0 \equiv \xi$. Replace v_t and $\xi_t(u)$ by ξ in the result of (b) (this is the first-order approximation in η), compute C_T in Equation (19.9), and show that the first-order skew is $\frac{\rho\eta}{2} E_H T^{H - 1/2}$ for every $T > 0$.

(d) In Bergomi's one-factor model, $\mathrm{d}\xi_t(u) = \xi_t(u) \, \eta e^{-\kappa(u-t)} \mathrm{d}W_t$ with $\kappa > 0$, the rest being as in rough Bergomi. Compute the first-order skew, its limit as $T \downarrow 0$ and its behaviour as $T \to \infty$. Compare with Theorem 19.11.

Exercise 19.6 *(Increments of the Riemann–Liouville process)* [⋆⋆⋆] Let Y be the Riemann–Liouville process of Theorem 19.5, $\alpha = H - \frac{1}{2}$ with $H \neq \frac{1}{2}$, and c_H as in Theorem 19.2.

(a) Show that for $s \geq 0$ and $h > 0$, $\mathbb{E}[(Y_{s+h} - Y_s)^2] = h^{2H}\Big(1 + 2H\int_0^{s/h}\big((1+x)^\alpha - x^\alpha\big)^2 \mathrm{d}x\Big)$.

(b) Deduce that this is strictly increasing in s, equals h^{2H} at $s = 0$ and tends to $2Hc_H^2 h^{2H}$ as $s \to \infty$. Does Y have stationary increments?

(c) Let B be a two-sided Brownian motion and $Z_t = \int_{-\infty}^0 \big((t-u)^\alpha - (-u)^\alpha\big)\mathrm{d}B_u$, $t \geq 0$. Show that for $0 < s < t$, $\mathbb{E}[(Z_t - Z_s)^2] \leq \frac{\alpha^2}{2-2H}\, s^{2H-2}(t-s)^2$, and that $(Z_{t+\varepsilon} - Z_t)/\varepsilon$ converges in L^2 as $\varepsilon \to 0$ for every $t > 0$.

(d) With $W = B$ on $[0,\infty)$, $c_H W_t^H = Y_t/\sqrt{2H} + Z_t$ defines a fBM (Theorem 19.2). Show that for fixed $s > 0$, $\big\|c_H(W^H_{s+h} - W^H_s) - (Y_{s+h} - Y_s)/\sqrt{2H}\big\|_{L^2} = O(h)$ as $h \downarrow 0$, while both increments have L^2 norm of order h^H. What does this say about the roughness of Y?

Exercise 19.7 *(The mean variance in rough Heston)* [⋆⋆⋆] In the rough Heston model of Theorem 19.15, with $a = H + \frac{1}{2} \in (\frac{1}{2}, 1]$, let $m(t) = \mathbb{E}^{\mathbb{Q}}[V_t]$. You may assume that $\sup_{t \leq T} \mathbb{E}^{\mathbb{Q}}[V_t] < \infty$ for every T, that the stochastic integral in Equation (19.11) has mean zero, and that $I^b I^c = I^{b+c}$ for $b, c > 0$.

(a) Show that $m(t) = V_0 + \lambda\, I^a[\theta - m](t)$ for $t \geq 0$.

(b) Let $E_a(x) = \sum_{n \geq 0} x^n/\Gamma(an+1)$ (the Mittag-Leffler function). Show that $I^a[s^{an}](t) = \frac{\Gamma(an+1)}{\Gamma(an+a+1)} t^{a(n+1)}$, and deduce that $\varphi(t) = E_a(-\lambda t^a)$ satisfies $\varphi = 1 - \lambda I^a \varphi$.

(c) Show that $m(t) = \theta + (V_0 - \theta)E_a(-\lambda t^a)$, by checking that it solves the equation of (a) and that this equation has at most one solution that is bounded and measurable on each $[0, T]$.

(d) Show that $m(t) = V_0 + (\theta - V_0)\lambda t^a/\Gamma(a+1) + O(t^{2a})$ as $t \downarrow 0$, and recover Theorem 15.11(i) when $a = 1$. How fast does the mean variance move away from V_0 when $H = 0.1$?

Exercise 19.8 *(The GJR regression under mean reversion)* [⋆⋆] This is a computational exercise; you may use `rfsv_logvol` and `gjr_estimate` from `ieor222.roughvol`, or your own code.

(a) Simulate daily values of the RFSV log-volatility Equation (19.5) with $H = 0.14$, $\nu = 0.3$, $m = -5$ for 3500 days, for $\lambda = 5 \times 10^{-4}$, 0.01 and 0.05 per day. Estimate H by the GJR procedure with $\Delta = 1, \ldots, 30$ and with $\Delta = 1, \ldots, 250$.

(b) Explain the pattern with Theorem 19.7(ii). Is the bound Equation (19.6) sharp?

(c) For $\lambda = 5 \times 10^{-4}$ and $\Delta = 1, \ldots, 30$, repeat with 20 independent paths and report the mean and standard deviation of $\hat{H}$. How precisely does one path of fourteen years determine H?

Chapter 20

Signatures and signature-based models

Exercise 20.1 *(Two corners)* [⋆] Let X be the piecewise linear path $(0,0) \to (1,0) \to (1,1)$ in $\mathbb{R}^2$, and Y the path $(0,0) \to (0,1) \to (1,1)$, each segment run in one unit of time.

(a) Compute levels 1 and 2 of $\mathrm{Sig}(X)_{0,2}$ and $\mathrm{Sig}(Y)_{0,2}$ with Theorem 20.6.

(b) Compute the two Lévy areas and interpret their values and signs as areas.

(c) Check the shuffle identity (Theorem 20.7) for $e_1 \sqcup\!\sqcup e_2$ and $e_1 \sqcup\!\sqcup e_1$ on both paths.

(d) Show that no linear functional of the signature truncated at level 1 distinguishes X from Y, and give one of level 2 that does.

Exercise 20.2 *(One dimension, and why time is added)* [⋆]

(a) Let $X : [0,T] \to \mathbb{R}$ be piecewise C^1, so that $\mathrm{Sig}(X)_{0,T} = \exp(X_T - X_0)$ by Theorem 20.7. If $X_T = X_0$, show directly that X is tree-like, with the height function $h(u) = |X_u - X_0|$, as Theorem 20.8 predicts. Give two paths with the same signature but different time averages $\frac{1}{T}\int_0^T X_t \,\mathrm{d}t$, and two with different maxima.

(b) Let $\widehat{X}_t = (t, X_t)$, with letters 0 (time) and 1. Show that $\int_0^T X_t \,\mathrm{d}t = TX_0 + \langle e_{10}, \mathrm{Sig}(\widehat{X})_{0,T}\rangle$, that $\langle e_{01}, \mathrm{Sig}(\widehat{X})_{0,T}\rangle = \int_0^T t \,\mathrm{d}X_t$, and that $\langle e_{01} + e_{10}, \mathrm{Sig}(\widehat{X})_{0,T}\rangle = T(X_T - X_0)$.

(c) On the set $\mathcal{P}$ of Theorem 20.10 with $d = 1$, show that $M(X) = \max_{t \le T} X_t$ satisfies $|M(X) - M(Y)| \le \delta(X,Y)$. What does Theorem 20.10 then say about M?

Exercise 20.3 *(Reversal and inverses)* [⋆⋆]

(a) Let $a \in T((\mathbb{R}^d))$ with $a^{\emptyset} = 1$, and $x = a - 1$. Show that $b = \sum_{k \ge 0} (-x)^{\otimes k}$ is well defined (each coordinate is a finite sum) and that $a \otimes b = b \otimes a = 1$. Write levels 1 and 2 of b in terms of those of a.

(b) Show that $\exp(v) \otimes \exp(-v) = 1$ for $v \in \mathbb{R}^d$. Deduce that for a piecewise linear path X on

$[0,T]$, its reversal $\overleftarrow{X}_u = X_{T-u}$ satisfies $\mathrm{Sig}(\overleftarrow{X})_{0,T} = \mathrm{Sig}(X)_{0,T}^{-1}$, and that X followed by $\overleftarrow{X}$ has signature 1.

(c) Check (a) and (b) at level 2 on the path X of Exercise 20.1, computing the signature of its reversal directly.

(d) Which statement of Theorem 20.8 does (b) illustrate? Give a height function h showing that X followed by $\overleftarrow{X}$ is tree-like, for the path X of Exercise 20.1.

Exercise 20.4 *(Lévy area of Brownian motion)* [⋆⋆] Let W^1, W^2 be independent Brownian motions, and $\mathcal{A}_T = \frac{1}{2}\int_0^T (W_u^1\,\mathrm{d}W_u^2 - W_u^2\,\mathrm{d}W_u^1)$ (Itô integrals).

(a) Show that the same expression with Stratonovich integrals, defined as in Section 20.2, is equal to $\mathcal{A}_T$.

(b) Show that $(\mathcal{A}_t)$ is a martingale with $\mathbb{E}[\mathcal{A}_T] = 0$ and $\mathbb{E}[\mathcal{A}_T^2] = T^2/4$. *Hint:* compute $\mathbb{E}\big[\int_0^T W^1\,\mathrm{d}W^2 \int_0^T W^2\,\mathrm{d}W^1\big]$ along the left-point sums of Theorem 7.9.

(c) Using Brownian scaling (Theorem 5.9) and the left-point sums of Theorem 7.9, show that $\mathcal{A}_T$ has the law of $T\mathcal{A}_1$. Check that this is consistent with (b).

Exercise 20.5 *(When is a linear signature model a martingale?)* [⋆⋆] Let $d = 1$, $\widehat{X}_t = (t, W_t)$ with letters 0 and 1, $\mathbb{X} = \mathrm{Sig}(\widehat{X})$ as in Theorem 20.12, and $S_t = \ell_\emptyset + \sum_{0<|I|\le 2} \ell_I \mathbb{X}_{0,t}^I$, the linear signature model with $n = 2$ (words $0, 1, 00, 01, 10, 11$).

(a) Express the six coordinates $\mathbb{X}_{0,t}^I$ in terms of t, W_t, $\int_0^t W_u\,\mathrm{d}u$ and $\int_0^t u\,\mathrm{d}W_u$.

(b) Show that S is a martingale if and only if $\ell_{00} = \ell_{10} = 0$ and $\ell_0 + \frac{1}{2}\ell_{11} = 0$. *Hint:* write S as an Itô process and compute $\mathbb{E}[S_t - S_s \mid \mathcal{F}_s]$. Check that this is the form of Theorem 20.15.

(c) In the martingale case, write $S_T - S_0 = \int_0^T (a + bu + cW_u)\,\mathrm{d}W_u$ and compute $\mathrm{Var}(S_T)$ with the Itô isometry. Then compute it again as $\theta^\top M\theta - S_0^2$ with Theorem 20.16(i), and compare.

Exercise 20.6 *(Variance swap and VIX in a signature volatility model)* [⋆⋆] Let $d = 1$ and, in the signature volatility model of Theorem 20.14(ii), let $\sigma_t = a + bt + cW_t$, that is $\ell = a\,e_\emptyset + b\,e_0 + c\,e_1$, with $b \neq 0$.

(a) Write $\sigma_t^2 = \langle \ell \sqcup\!\sqcup \ell, \mathbb{X}_{0,t}\rangle$ explicitly as a combination of words.

(b) Compute the variance-swap strike $\frac{1}{T}\mathbb{E}^{\mathbb{Q}}\int_0^T \sigma_t^2\,\mathrm{d}t$ directly, and with Theorem 20.16(ii) and the expected signature of Theorem 20.13(iv).

(c) Compute $\mathbb{E}^{\mathbb{Q}}\big[\int_T^{T+\Delta} \sigma_u^2\,\mathrm{d}u \mid \mathcal{F}_T\big]$ directly, writing $W_u = W_T + (W_u - W_T)$.

(d) Check the terms in c^2 of (c) against Theorem 20.16(iii): compute $Q_{11}(T) = \langle (e_1 \sqcup\!\sqcup e_1) \otimes e_0, \mathbb{X}_{0,T} \otimes (E_\Delta - 1)\rangle$ with Chen's rule for products.

Exercise 20.7 *(Black–Scholes as a signature model)* [⋆⋆⋆] Let $S_t = S_0 \exp(\sigma W_t - \frac{1}{2}\sigma^2 t)$ with $S_0, \sigma > 0$, and define iterated Itô integrals by $J_0 = 1$ and $J_k(t) = \int_0^t J_{k-1}(u)\,\mathrm{d}W_u$. Let the polynomials $H_k(x,t)$ be defined by $\exp(\lambda x - \frac{1}{2}\lambda^2 t) = \sum_{k\ge 0} \lambda^k H_k(x,t)/k!$ for all real λ.

(a) Show that $\partial_x H_k = kH_{k-1}$ and $\partial_t H_k + \frac{1}{2}\partial_{xx} H_k = 0$, and deduce with Itô's formula that $J_k(t) = H_k(W_t, t)/k!$.

(b) Show that $\mathbb{E}[J_k(t)J_l(t)] = \mathbf{1}_{\{k=l\}}\, t^k/k!$.

(c) Show that $S_T = S_0 \sum_{k\geq 0} \sigma^k J_k(T)$ in L^2, and that the truncation $S_T^{(n)} = S_0 \sum_{k\leq n} \sigma^k J_k(T)$ satisfies $\mathbb{E}[(S_T - S_T^{(n)})^2] = S_0^2 \sum_{k>n} \sigma^{2k} T^k/k!$.

(d) With Theorem 20.13(iii), write J_1, J_2, J_3 as linear functionals of $\mathbb{X}_{0,t}$, letters 0 and 1. Explain why $S^{(n)}$ is a linear signature model in the martingale form of Theorem 20.15, and give its coefficients ℓ_I^1 for $n = 3$. For $\sigma = 0.2$ and $T = 1$, compute the relative error $\|S_T - S_T^{(n)}\|_{L^2}/\|S_T\|_{L^2}$ for $n = 1, 2, 3$.

Exercise 20.8 *(Calibrating to a flat smile)* [⋆⋆] This is a computational exercise; you may use `ieor222.signatures` and `ieor222.volatility`. Take Black–Scholes call prices with $\sigma = 20\%$, $r = 0$, $S_0 = 1$, maturities $T \in \{0.25, 1\}$ and strikes $K = e^{0.2\sqrt{T}x}$, $x \in \{-1.5, -0.75, 0, 0.75, 1.5\}$, and the linear signature model in martingale form driven by one Brownian motion, with primary process (t, W_t).

(a) Calibrate the model with $n = 1$, $S_t = 1 + \ell_\emptyset^1 W_t$, on 40,000 antithetic paths, and report the implied-volatility errors. Explain their shape. (Compare with Bachelier's model, Theorem 15.5.)

(b) Repeat with $n = 2$ and $n = 3$, reporting the errors on the calibration paths and on 40,000 independent paths. Compare the calibrated coefficients with the truncation of Exercise 20.7.

(c) Which value of n would you use, and why?